Solutions for LAMC Monthly Contest

Advanced Group

December 9, 2012

1. If $a \geq 3$, then
$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c} < 3 < \overline{a,b(c)}$$
contradiction. If $a \geq 2$ and b or c is greater or equal then 2, then
$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c} \leq 2 < \overline{a,b(c)}$$
contradiction. If $a = 2$ and $b = c = 1$, then we get again a contradiction. If $a = 1$, then we get that
$$\frac{1}{b}+\frac{1}{c} = \overline{0,b(c)} = \frac{b}{10}+\frac{c}{90}$$
so $90(b+c) = bc(9b+c)$. If $3 \nmid c$, then $9|bc$ so $b = 9$, which is not a solution. If $c = 3$, then we get $b = 5$. If $c = 6$ or $c = 9$, we get no solutions. Hence the only solution is $(a,b,c) = (1,5,3)$.

2. a). As $PA^2 + PC^2 = PB^2 + PD^2$, hence $PD = \sqrt{2}$.

b). As $AB = AD$, $PB = PD$, and AP is a coomon side, we get that the triangles $\triangle APB$ and $\triangle APD$ are congruent, therefore $\angle DAP = \angle PAB = 45\,\text{deg}$. Let M be the projection of P on AB. The triangle $\triangle PAM$ is a right isosceles triangle, so $PM = 1/\sqrt{2}$, and as $PB = \sqrt{2} = 2PM$, then $\angle PBM = 30°$. So $\angle APB = 105°$.

3. The last result of Tom should be an even number from $\{978, \ldots, 999\}$, these numbers are $978, 980, 982, 984, 986, 988, 990, 992, 994, 996, 998$. If he got $978, 982, 986, 990, 994, 998$, these are obtained by the doubling of $489, \ldots, 499$, numbers that cannot be obtained during the game as the numbers obtained during the game are all even. For the other numbers we have the following possible plays of Tom and Bill:

$980 \leftarrow 490(Tom) \leftarrow 468 \leftarrow 234(Tom) \leftarrow 212 \leftarrow 106(Tom) \leftarrow 84 \leftarrow 42(Tom) \leftarrow 20 \leftarrow 10(Tom)$

$984 \leftarrow 492(Tom) \leftarrow 470 \leftarrow 235(Tom)$

$988 \leftarrow 494(Tom) \leftarrow 472 \leftarrow 236(Tom) \leftarrow 214 \leftarrow 107(Tom)$

$992 \leftarrow 496(Tom) \leftarrow 474 \leftarrow 237(Tom)$

$996 \leftarrow 498(Tom) \leftarrow 476 \leftarrow 238(Tom) \leftarrow 216 \leftarrow 108(Tom) \leftarrow 86 \leftarrow 43(Tom)$.

The name Tom appears at the numbers that he can choose in order to win the game. Counting all of them, we get that Tom has 22 possibilities in order to win the game.

4.a). As $m^2-2n|n^2+2m$ and $n^2+2m>0$, we have that $m^2-2n \le n^2+2m$, so $m \le n+2$. Similarly, by the other condition, $n \le m+2$. Hence $|m-n| \le 2$.

b). Suppose that $n \ge m$. By the condition above, $n \in \{m, m+1, m+2\}$. If $n = m$, we get that

$$\frac{n^2+2n}{n^2-2n} = 1 + \frac{4}{n-2} \in \mathbb{Z}, \text{so } n \in \{1,3,4,6\}$$

If $n = m+1$, then

$$\frac{m^2+2m+2}{m^2+1} = 1 + \frac{2m+1}{m^2+1}, \text{so } m \le 2,$$

and we see that if $m = 2$, then $n = 3$, but the second fraction is not an integer. If $n = m+2$, the first fraction is 1, but the second fraction is

$$\frac{m^2+6m+4}{m^2-2m-4} = 1 + \frac{8m+8}{m^2-2m-4}$$

so $m^2-2m-4 \le 8m+8$, or equivalently $m^2-10m-12 \le 0$, implying that $m \in \{2,3,4,5,6,7\}$. The only solutions are $m=2$, $m=3$, $m=4$. Therefore, using the symmetry of the fractions we get that

$$(m,n) = (1,1),(3,3),(4,4),(6,6),(2,4),(4,2),(3,5),(5,3),(4,6),(6,4).$$

5. The condition translates to $pq = p+q = a$. As $(p+q)^2 \ge 4pq$, we get that $a \ge 4$. We have

$$\begin{aligned}\frac{1}{p(p+1)} + \frac{1}{q(q+1)} &= \frac{p^2+q^2+p+q}{pq(p+1)(q+1)} = \frac{(p+q)^2-2pq+p+q}{pq(pq+p+q+1)} \\ &= \frac{a^2-2a+a}{a(a+a+1)} = \frac{a^2-a}{a(2a+1)} = \frac{a-1}{2a+1}\end{aligned}$$

and $\frac{a-1}{2a+1} < \frac{1}{2} \Leftrightarrow 2a-2 < 2a+1$, and $\frac{a-1}{2a+1} > \frac{1}{3} \Leftrightarrow a \ge 4$, which are both true. Also

$$\frac{1}{p(p-1)} + \frac{1}{q(q-1)} = \frac{p^2+q^2-p-q}{pq(p-1)(q-1)} = \frac{(p+q)^2-2pq-p-q}{pq(pq-p-q+1)} = \frac{a^2-3a}{a} = a-3 \ge 1.$$