RANDOM VARIABLES

MATH CIRCLE (ADVANCED) 3/3/2013

0)

a) Suppose you flip a fair coin 3 times.

i) What is the probability you get 0 heads?

ii) 1 head?

iii) 2 heads?

iv) 3 heads?

For $k = 0, 1, 2, 3$, $P(k \text{ Heads}) = \frac{\binom{3}{k}}{2^3}$

b) Suppose you are dealt 3 cards from a standard deck of cards.

i) What is the probability you get 0 hearts?

ii) 1 heart?

iii) 2 hearts?

iv) 3 hearts?

For $k = 0, 1, 2, 3$, $P(k \text{ Hearts}) = \frac{\binom{13}{k}\binom{39}{3-k}}{\binom{52}{3}}$

Notice how similar the answers are in i,ii,iii, and iv. Go back and modify your answers so that all the parts look almost identical.

1)

a) Consider the following game with John. You flip a fair coin once. If the flip is Heads, you give John $\$2$. If the flip is Tails, John gives you $\$1$. Would you play the game with John? Why or why not?

No. The expected winnings is $\frac{1}{2}(\$1) + \frac{1}{2}(-\$2) = -\$.5$.

b) Suppose the game is changed so that in either outcome, $\$1$ is exchanged. Now would you play the game with John? Why or why not?

Probably, the game is fair. The expected winnings is $\frac{1}{2}(\$1) + \frac{1}{2}(-\$1) = \$0$. The variance in the winnings $\frac{1}{2}(1)^2 + \frac{1}{2}(-1)^2 = 1$.

c) Finally, suppose the game is changed so that on Heads, you give John $\$1000$, while on Tails, John gives you $\$1000$. Would you play the game with John? Why or why not? What is similar/different to the game in b)?

Probably not, but the game is fair. The expected winnings is $\frac{1}{2}(\$1000)+\frac{1}{2}(-\$1000) = \$0$. The variance in the winnings $\frac{1}{2}(1000)^2 + \frac{1}{2}(-1000)^2 = 1000000$, so the game has much more variance.

Random Variable (Informal): A *random variable* is a quantifiable experiment. That is, an experiment with numerical outcomes. Random variables are usually denoted by capital letters (X, Y, Z,etc.).

For example, in question 0 we could define random variables X =the number of heads in part a), or Y =the number of hearts in part b).

Random Variable (Formal): A *random variable* X is a collection of numbers, called the *range* of X (denoted range(X)) along with probabilities associated with each member of the range, satisfying the following

i) For $k \in \text{range}(X)$, $0 \leq P(X = k) \leq 1$.

ii) $\sum_{k\in\text{range}(X)} P(X = k) = 1$.

The function $f_X(k) = P(X = k)$ is called the *probability mass function* (PMF).

2) Suppose you flip a coin 10 times. Let X =the number of heads.

a) What is the range of X?

The range is $k = 0, 1, 2, \ldots, 10$.

b) What is the PMF of X? (Hint: Think of 0a.)

For k in the range: $P(X = k) = \frac{\binom{10}{k}}{2^{10}}$.

3) Suppose you are dealt 8 cards from a deck of cards. Let Y =the number of hearts.

a) What is the range of Y?

The range is $k = 0, 1, 2, \ldots, 8$.

b) What is the PMF of Y?

For k in the range: $P(Y = k) = \frac{\binom{13}{k}\binom{39}{3-k}}{\binom{52}{3}}$.

Sometimes it is easier to write the PMF as a chart. Do so in the following problem.

4) Xavier and Yuri each pick a number from $1, 2, 3$ at random (they could both pick the same number). Let X = Xavier's number and Y = Yuri's number.

a) Find the PMF of X and the PMF of Y

k	P(X=k)	P(Y=k)
1	$\frac{1}{3}$	$\frac{1}{3}$
2	$\frac{1}{3}$	$\frac{1}{3}$
3	$\frac{1}{3}$	$\frac{1}{3}$

b) Define new random variables $Z = X + Y$ and $W = |X - Y|$. Explain what Z and W are in your own words.

Z is the sum of the number, while W is the difference.

c) Find the PMF of Z and the PMF of W.

k	P(Z=k)
2	$\frac{1}{9}$
3	$\frac{2}{9}$
4	$\frac{3}{9}$
5	$\frac{2}{9}$
6	$\frac{1}{9}$

k	P(W=k)
0	$\frac{3}{9}$
1	$\frac{4}{9}$
2	$\frac{2}{9}$

As we saw in question 1, there are other questions we can ask about random variables.

Expected Value: The *expected value* (denoted EX) or mean of a random variable X is given by the equation

$$EX = \sum_{k \in \text{range}(X)} k \cdot P(X = k).$$

If we performed our experiment (X) many times and averaged all the results together, we would most likely get a value close to EX.

Variance: The *variance* (denoted $\text{Var}(X)$) of a random variable X is given by the equation

$$\text{Var}(X) = \sum_{k \in \text{range}(X)} (k - EX)^2 P(X = k).$$

The variance measures, on average, how far away a random variable is away from its mean.

5) Modify your answer to 1 by calculating the expected value and variance for each game.

See 1).

6) Suppose you have a coin with $P(\text{heads}) = p$. Flip the coin once, and let $X =$the number of heads.

a) Find the PMF of X.

$P(X = 1) = p, P(X = 0) = (1 - p)$.

b) Find EX.

$EX = 1 \cdot p + 0 \cdot (1 - p) = p.$

c) Find $\text{Var}(X)$.

$\text{Var}(X) = (1 - p)^2 \cdot p + (-p)^2 \cdot (1 - p) = (1 - p) \cdot p[(1 - p) + p] = p(1 - p).$

7) Suppose we have a random variable X with PMF.

k	P(Y=k)
-3	.1
-1	.2
1	.3
2	.3
6	.1

Find EX and $\text{Var}(X)$.

$EX = (-3)(.1) + (-1)(.2) + (1)(.3) + (2)(.3) + (6)(.1) = 1.$

$\text{Var}(X) = (-3 - 1)^2(.1) + (-1 - 1)^2(.2) + (1 - 1)^2(.3) + (2 - 1)^2(.3) + (6 - 1)^2(.1) = 5.2.$

8) Recall X, Y, Z from problem 4.

a) Calculate $EX = EY$ and EZ.

$EX = EY = \frac{1}{3}(1+2+3) = 2.$

$EZ = 2 \cdot \frac{1}{9} + 3 \cdot \frac{2}{9} + 4 \cdot \frac{3}{9} + 5 \cdot \frac{2}{9} + 6 \cdot \frac{1}{9} = 4.$

b) Do you notice any relationship between the three?

$EX + EY = EZ.$

Independence (Informal): Two events A, B are independent if knowing one happened doesn't affect the probability of the other, i.e.

$$P(A|B) = P(A) \text{ and } P(B|A) = P(B).$$

Independence (Formal): Two events A, B are independent if $P(A \cap B) = P(A) \cdot P(B)$.

9) Prove that if A, B are independent in the informal sense then they are independent in the formal sense.

$P(A \cap B) = P(A)P(B|A) = P(A)P(B)$ as needed.

10) Suppose A, B are events such that $P(A) = .3, P(A^c \cap B^c) = .2, P(B^c) = .4$. Are A and B independent?

We know $P(B) = .6$. Further, $P(A^c \cap B^c) = 1 - P(A \cup B) = .2$, so $P(A \cup B) = .8$. Thus, $P(A \cap B) = P(A) + P(B) - P(A \cup B) = .1$. Since $.1 \neq .3 \cdot .6$, A, B are not independent.

11) Suppose A, B are disjoint events. Are A, B independent? Why or why not?

Note if $P(A), P(B) > 0$, then $0 = P(\emptyset) = P(A \cap B) \neq P(A)P(B)$. Therefore, as long as $P(A), P(B) > 0$ then A, B are NOT independent. If $P(A)$ or $P(B)$ is zero, then they are independent.

12) Suppose a family has kids. Let $A =$ at least one boy and at least one girl. Let $B =$ at most one girl. Are A and B independent:

a) If the family has two kids?

In this case $A = 1B, 1G$, $B = \leq 1G$, and $A \cap B = 1B, 1G$. Thus, $\frac{2}{4} = P(A \cap B) \neq P(A)P(B) = \frac{2}{4} \cdot (1 - \frac{1}{4}$

b) If the family has three kids?

In this case $A = 2B, 1G$ or $1B, 2G$, $B = \leq 1G$, and $A \cap B = 2B, 1G$. Thus, $\frac{3}{8} = P(A \cap B) = P(A)P(B) =$

13) Suppose an urn contains 4 red, 5 green, and 5 yellow balls. You pick one ball, note it, and replace it (back in the urn). If you do this a total of 6 times, what is the probability:

a) you get 3 green balls?

$\binom{6}{3}(\frac{5}{14})^3(\frac{9}{14})^3$

b) you get 3 green and 2 red balls?

$\frac{6!}{3!2!1!}(\frac{4}{14})^2(\frac{5}{14})^3(\frac{5}{14})^1$

Properties of Expected Value: If a and b are constants and X and Y are random variables, then:

i) $E(aX + b) = aE(X) + b$

ii) $E(X + Y) = EX + EY$

14) (Binomial Distribution) Suppose you have a coin with $P(\text{heads}) = p$. Flip the coin n times and let $X =$ the number of heads.

a) Find the PMF of X.

For $k = 0, 1, 2, \ldots, n$: $P(X = k) = \binom{n}{k} p^k (1-p)^{n-k}$.

b) Find EX. Hint: use property ii above along with question 6.

Let X_i =the number of heads on the ith flip. Then $X = X_1 + \cdots + X_n$ so $EX = EX_1 + \cdots + EX_n = p + \cdots p = np$.

15) Suppose you pay $\$M$ to play a game. The game consists of a coin (with $P(\text{heads}) = 1/6$) being flipped 30 times. You win $\$4$ for each heads. For what values of M would you play the game? Hint: Use property i above along with the previous question.

Let Y be the winnings and X be the number of heads. Then $Y = 4X - M$. By the previous problem $EX = 30 \cdot \frac{1}{6} = 5$, so $EY = 4 \cdot 5 - M$, We want $EY \geq 0$ if we play the game, so $M \leq \$20$.

16)* (Geometric Distribution)

Suppose you have a coin with $P(\text{heads}) = p$. Flip the coin *until* you get a heads. Let X =the number of flips.

a) Find the PMF of X. Hint: Start by doing a few examples.

For $k = 1, 2, 3, \ldots$: $P(X = k) = (1-p)^{k-1} p$

b) Argue that $EX = 1/p$.

Hint: Suppose you flipped the coin EX times. What should the expected number of heads be?

If we flip the coin EX times, the problem 10 tells us we should expect $EX \cdot p$ heads. However, this should also be one, as we flip the coin until the first head. Thus, $EX = 1/p$.