ORMC Intermediate 2: Projective Geometry

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1 Review

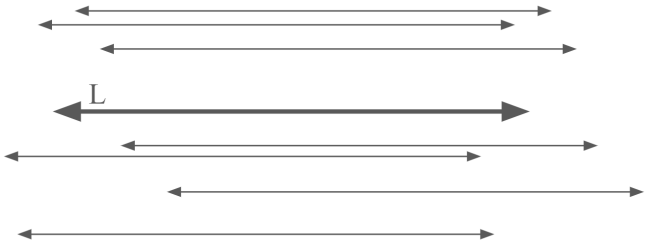
Railroad tracks are parallel, meaning that they never intersect, but they appear to converge in the far distance. Why is this? Recall that we can use **projections** to map images from three-dimensional space onto a two-dimensional plane. The projected image shows how the original image is seen from the point at which it is observed, which we can call the *center of projection*. When viewing railroad tracks, the three dimensional landscape in front of us is projected onto the two-dimensional plane of our retinas in the back of our eyes. The point at which the tracks appear to converge is the *vanishing point* at infinity. Projection is also what allows us to create drawings and paintings which appear realistic.

Figure 1: "The Concert" by Vermeer appears more realistic through the use of perspective. The square tiles on the floor are elongated to give the perception of depth.

*With much help from Wikipedia and Wolfram MathWorld

2 One-, Two-, and Three-Point Perspective

Let's define a **pencil** as a family of geometric objects that share a common property. For example, we can define a pencil of lines parallel to the line L, below. All lines parallel to L are in the pencil, but only some of these lines are shown.



You can create a three-dimensional solid from the intersections of three pencils of parallel lines: a "width" pencil (side-to-side), a "height" pencil (up and down), and a "length" pencil (front-to-back).

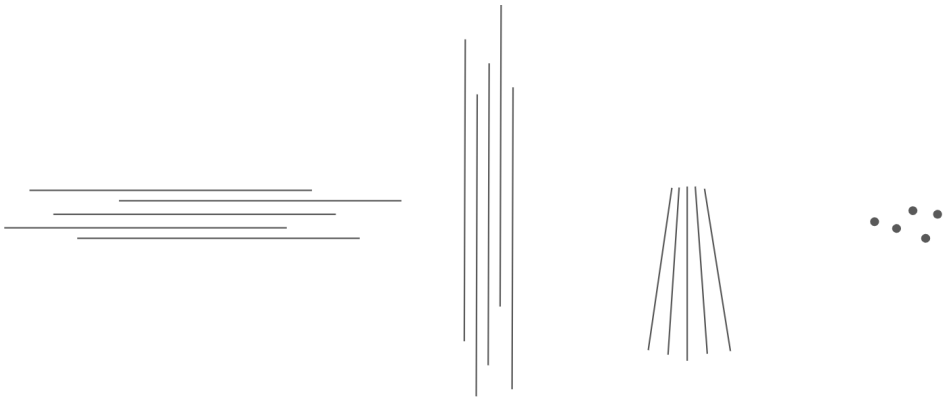


Figure 2: Three dimensions of pencils: width, height, and length. Only five objects in each pencil are shown, but a pencil contains all of the objects with a common property (all lines parallel to the lines shown). Notice how perspective is necessary to draw the *length* pencil at all: without it, these parallel lines would go directly into the page and appear as dots (shown to the right).

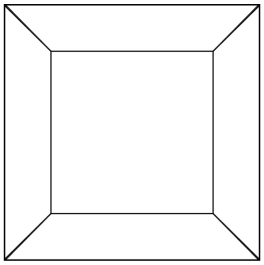
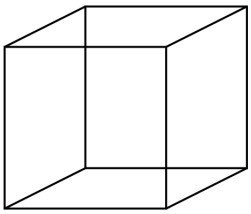


Figure 3: The three pencils shown above could be used to draw a cube like this one.

1. Label the lines used to draw the cube above as belonging to the width, height, or length pencils defined previously.

One-Point Perspective

Imagine that you want to draw a three-dimensional solid so that it looks more realistic. You could take a cube such as the one shown below and redraw it so that when viewed along a specific direction, the nearer side appears larger than the farther side, the way that objects appear in real life.



2. Redraw the cube above using one-point perspective: Define the "width," "height," and "length" pencils for this drawing. Leave the "height" and "length" pencils as they are. Choose a vanishing point V for the "width" pencil: all the parallel lines from the "width" pencil, if extended beyond the edges of the cube, intersect at V.

Two-Point Perspective

3. Redraw the cube so that the "width" and "length" lines meet at finite points V and W.

Three-Point Perspective

4. Redraw the cube so that the "width," "length," and "height," lines all meet at finite points: V, W, U.

3 Projective Geometry

Let's define a **projection**: a transformation of points and lines from 2D or 3D space onto another 2D plane, which can be achieved by connecting corresponding points on the two planes with parallel lines.

Projective geometry is the branch of geometry which studies geometric properties that are *invariant* - they remain the same after undergoing a projective transformation. You are most likely used to Euclidean geometry from math classes at school. Compared to Euclidean geometry, projective geometry has more points and can use geometric transformations to turn these "points at infinity" into Euclidean points. Projective geometry is also considered *non-metrical*, meaning that distances are not meaningful or conserved.

Projective geometry can be defined by the following axioms:

(a) If A and B are distinct points on a plane, there exists exactly one line containing both A and B.

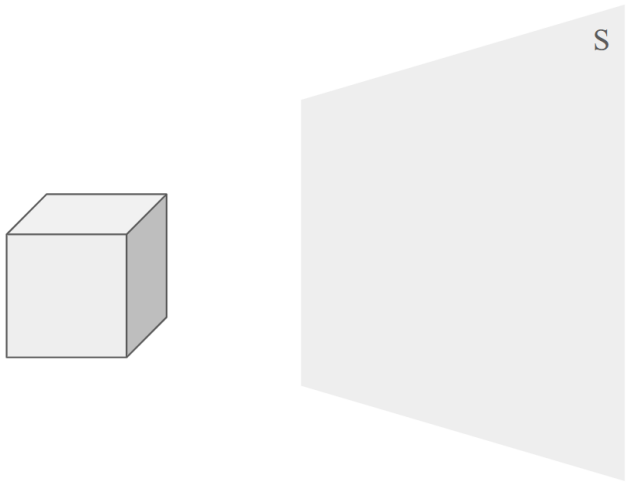
(b) Any two lines in a plane have at least one point of the plane in common. This point can be the point at infinity. (We will go further into this later.)

(c) There is at least one line on a plane.

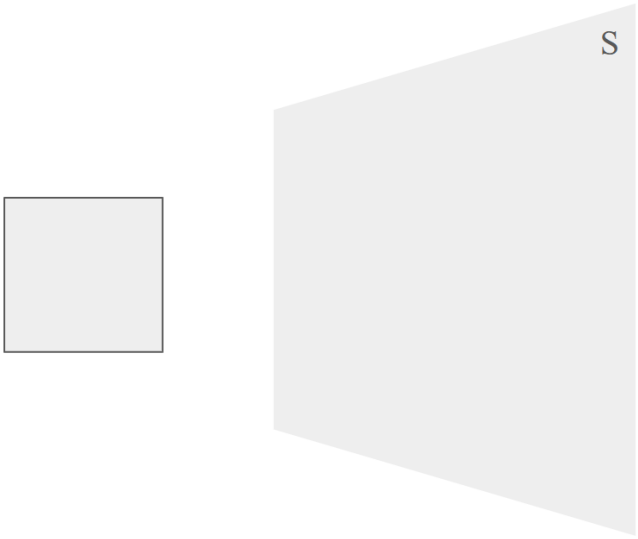
(d) Every line contains at least three points of the plane.

(e) All the points of the plane do not belong to the same line.

5. Consider: Parallel lines appear to converge at a point "at infinity" (recall the train tracks). How might projective geometry turn a point "at infinity" into a vanishing point like the ones used to draw cubes in perspective on the previous page?



6. Project the cube above onto the screen S: Define an observation point O on the opposite side of S from the cube. Draw lines from the corners of the cube to O. Where the lines pass through S, mark the projected locations of the cube corners. Fill in the cube's edges.

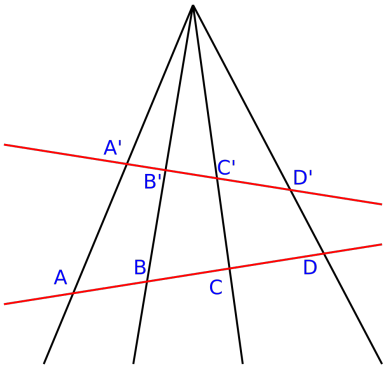


7. Project the square onto screen S. Convince yourself that angles are not invariant in projective geometry (they are not conserved across projections).

8. Consider: Do you think that straight lines are invariant in projective geometry? What about curves (i.e., does a circle always remain a circle)? We will go into this in more depth later.

Given four points on a line, their **cross ratio** is defined as:

$$R_c(A, B, C, D) = \frac{AC/BC}{AD/BD}$$



9. Can you demonstrate that $R_c(A, B, C, D) = R_c(A', B', C', D')$? If so, provide a proof or convincing argument.

This is known as the *invariance* of the cross ratio.

10. Will the invariance of the cross ratio remain if you project ABCD onto a different line? What if you use a different *center of projection* (point of origin for the lines which project each point A, B, C, D onto a new line)?

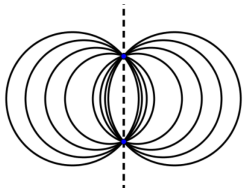
4 Projective Plane

In Euclidean geometry, the kind of geometry you learn in school, two lines on the same plane intersect at a single point unless they are parallel, in which case they do not intersect. A real projective plane (or "extended Euclidean plane") contains extra "points at infinity" where parallel lines intersect. You can visualize such a point by imagining a railroad track going off into the distance, where the parallel rails appear to converge into a vanishing point. At a distance of infinity, this vanishing point exists as a point in the projective plane where the parallel rails actually intersect.

A projective plane C consists of a set of points P, a set of lines L, and a symmetric relation I on the set $P \cup L$, which is known as the *incidence*. Incidence describes a symmetric relationship between two geometric entities that interact, i.e. a point P is incident to a line l if P lies on l, which is the same as l passing through P.

11. In our last packet discussing projections, we claimed that parallel lines never intersect. Now we claim that parallel lines in the same plane intersect at infinity. Why is this not actually a contradiction?

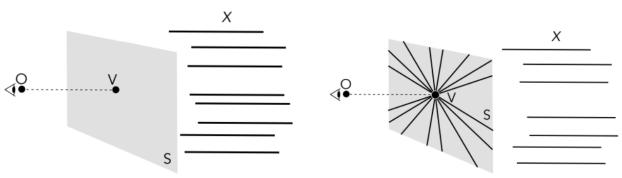
Recall that a **pencil** in geometry is a family of geometric objects with a common property. For example, a pencil of points is the set of all points on a given line and a pencil of circles can be all the circles in the same plane which pass through a specified pair of points.



12. In the Euclidean plane, we can define a pencil, or set, of parallel lines. What is their common property in the projective plane?

13. How many points at infinity define a class (unique set) of mutually parallel lines?

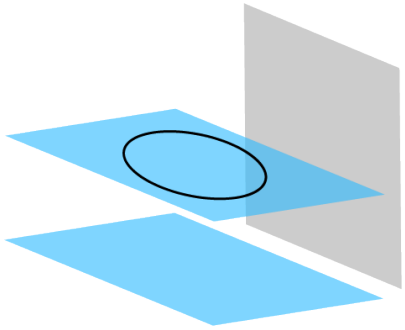
14. Use the concept of a pencil to explain why a set of parallel lines appear to intersect at a vanishing point, as seen below.



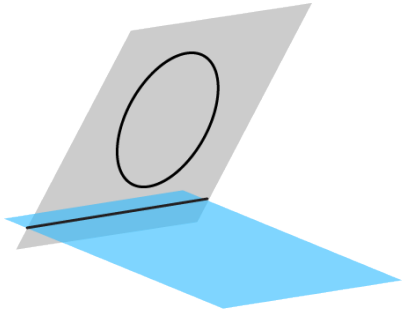
5 Conic Sections

A **conic section** is any curve you can get from a circle through a projective transformation. Let's look at some different types of conic sections.

15. If you have a circle with a plane parallel to it and a plane perpendicular to it, what shape do you get from each projection?

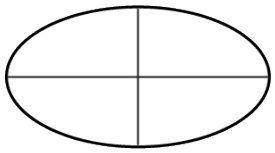


16. Project the circle below from one plane to another. What is the resulting shape?

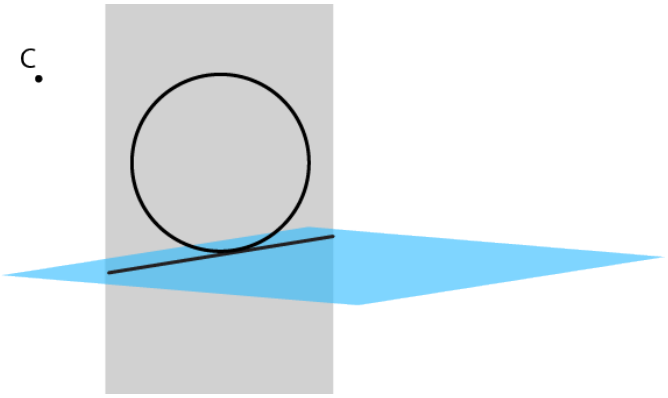


17. Recall that a diameter is any line passing through the center of a circle from one edge to the other. A circle has an infinite number of diameters. How many diameters does the projected circle have? Compared to the original circle, what is the largest possible diameter of the projected circle?

    For an ellipse, two diameters are considered important. The major axis is the _____ diameter, and the minor axis is the _____ diameter.

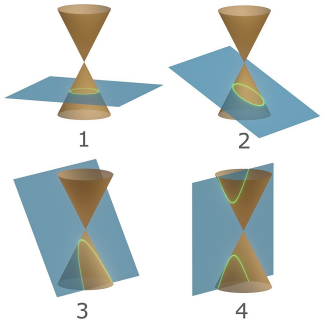


    The plane of the original circle and the plane onto which it was projected to form an ellipse meet in a line. Relative to this line, which diameters of the original circle form the major and minor axes of the ellipse?



18. Project the circle above from one plane to the other with the center of projection C located behind the original circle, on the opposite side from the plane onto which it is being projected. What shape is created?

The projections we have just created are all called **conic sections** because they are the result of the intersection of a plane with the surface of a double cone. You could define a cone as a set of all the parallel projections of a circle with respect to a single center of projection, which lies at the vertex of the cone. Below, we can see a circle, an ellipse, a parabola, and a hyperbola created from the intersection of different planes with the double cone.



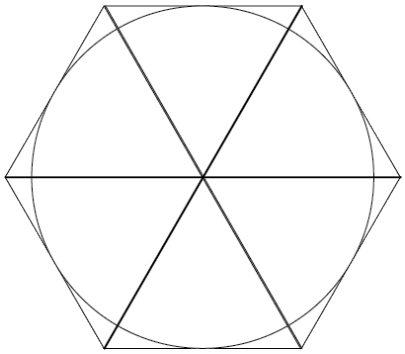
19. A hyperbola (shape 4) occurs when the plane intersects both sides of the double cone. How must the plane be oriented so that it creates a parabola (shape 3), not a hyperbola?

6 Duality

The **duality principle** states that in projective geometry, all propositions exist in pairs from which either member of the pair can be inferred from the other by swapping the words "point" and "line." Essentially, duality provides us with a way of mapping from one half of a dual pair to the other and back again. The simplest dual pair describes the relationship between points and lines: 1) any two distinct points define a unique line, and 2) any two distinct lines define a unique point (their point of intersection).

One interesting dual pair is Brianchon's Theorem and Pascal's Theorem.

Brianchon's Theorem states that for a hexagon circumscribed around a conic section, all of the principal diagonals (lines connecting opposite vertices) meet at a single point.



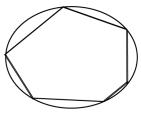
20. The hexagon above is regular. Does Brianchon's theorem apply to irregular circumscribed hexagons? Explain or demonstrate.

21. Can Brianchon's theorem be expanded to apply to any circumscribed polygons other than hexagons? What is the smallest polygon to which it applies? The largest?

Pascal's Theorem is the dual of Brianchon's theorem.

22. Pascal's theorem states that for a hexagon inscribed *in* a conic section (such as a circle or ellipse), continuations of the three pairs of opposite sides meet to create what kind of geometric entity? Draw a diagram demonstrating Pascal's theorem on the image below.

Hint: Recall the definition of the duality principle.



23. Does Pascal's theorem apply to regular hexagons? How about concave hexagons?

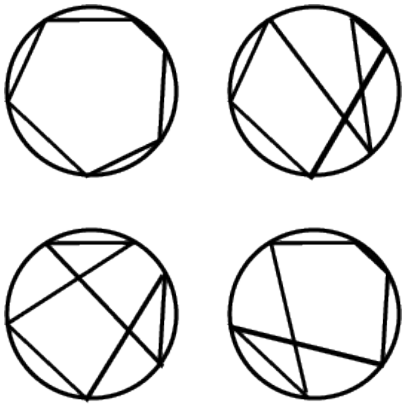
24. Can Pascal's theorem be expanded to apply to any circumscribed polygons other than hexagons? What is the smallest polygon to which it applies? The largest?

Mobius published generalizations of both Brianchon's and Pascal's theorems.

25. Mobius' generalization of Brianchon's theorem states that if all principal diagonals except one of a certain category of polygons circumscribed on a conic section meet in one point, then the remaining principal diagonal will meet at that same point. For what category of polygons (how many sides) does this statement apply?

26. What is Mobius' generalization of Pascal's theorem?

Let us now introduce the full version of Pascal's theorem. Take a circle or ellipse (other conic sections work as well, but we will not deal with parabolas or hyperbolas today). Choose six arbitrary points on the outside of the circle or ellipse. Connect these points in any order to form a hexagon. Notice that the sides of the hexagon may intersect (cross) one another.



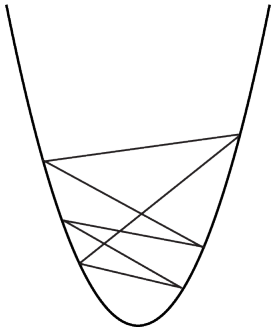
27. How many different hexagons can you create in this way?

A **Pascal line** is the line created by extending pairs of opposite sides of a hexagon until they meet at points, then joining those three points along a line.

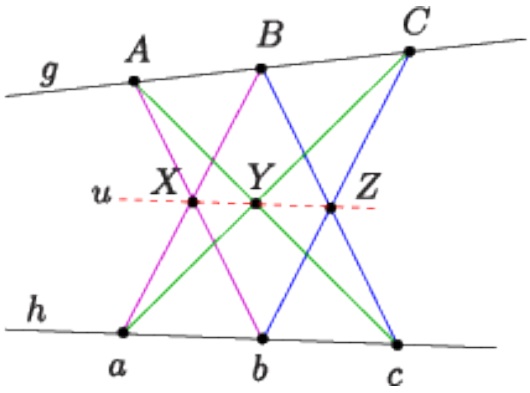
28. How many total unique Pascal lines can be created by connecting the vertices of a hexagon in any order?

29. Challenge: At how many points do these lines intersect? How many lines intersect at a time? *Hint: Consider the case of a regular hexagon inscribed in a circle.* (This problem is difficult. Feel free to move on to the next section if you spend some time on it without success.)

30. Given the hexagon inscribed in the parabola below, demonstrate that Pascal's theorem still applies. How can you use projections to demonstrate that Pascal's theorem holds for a hexagon inscribed in any conic section?



Pappus's hexagon theorem is a version of Pascal's theorem which eliminates the necessity of inscription within a conic section. It states that given a set of collinear points A, B, C on line g and another set of collinear points a, b, c on line h, the intersection points X, Y, Z of line pairs Ab and aB, Ac and aC, and Bc and bC are also collinear on a line known as the *Pappus line*, u.



31. What is the hexagon for which the line pairs described above form opposite sides? (A hexagon is defined by writing its six vertices in order.)

32. Demonstrate that Pappus's hexagon theorem holds in the case that lines g and h share a common point (that is not A, B, C, a, b, or c).

Let's construct the **dual theorem of Pappus** by swapping lines for points and vice versa. Follow the instructions below to draw a diagram of the dual theorem of Pappus:

(a) Lines g and h become points G and H, with lines A, B, C radiating out from G and lines a, b, c radiating out from H (choose angles between each set of lines of < 40 °for ease of viewing).

(b) Lines Ab, aB, Ac, aC, Bc, and cB now become points.

(c) $X := (A \cap b)(a \cap B)$

$Y := (c \cap A)(C \cap a)$

$Z := (b \cap C)(B \cap c)$

(d) What geometric entity does the line u from Pappus's theorem become in the dual of Pappus's theorem?