Classical Theorems and Problems in Euclidean Geometry

ORMC Math Circle

Theorems

In the first part of today, we will focus on proving the following classical theorems and developing the geometric techniques that connect them.

1. **Simson Line:** Let ABC be an acute triangle with circumcircle ω. Let P be a point on ω. Let D, E, and F be the feet of the perpendiculars from P to the lines BC, CA, and AB, respectively (so $D \in BC$, $E \in CA$, $F \in AB$).

   Prove that the points D, E, and F are collinear.

2. **Butterfly Theorem:** Let M be the midpoint of a chord $\overline{PQ}$ of a circle. Through M, draw two other chords $\overline{AB}$ and $\overline{CD}$. Let lines AD and BC intersect chord $\overline{PQ}$ at points X and Y, respectively.

   Prove that M is the midpoint of $\overline{XY}$.

3. **Nine-Point Circle:** Consider a triangle ABC with orthocenter H. Let D, E, and F be the feet of the altitudes from A, B, and C, respectively. Let M_A, M_B, and M_C be the midpoints of $\overline{BC}$, $\overline{CA}$, and $\overline{AB}$, respectively. Let M_{AH}, M_{BH}, and M_{CH} be the midpoints of $\overline{AH}$, $\overline{BH}$, and $\overline{CH}$, respectively, as shown.

   Prove that these nine points

   $$D,\ E,\ F,\ M_A,\ M_B,\ M_C,\ M_{AH},\ M_{BH},\ M_{CH}$$

   are concyclic; the resulting circle is called the Nine-Point Circle.

4. **Euler Line Theorem:** Let ABC be a non-equilateral triangle. Let G be its centroid, O its circumcenter, and H its orthocenter.

   Prove that the points O, G, and H are collinear. Moreover, show that G lies between O and H and that

   $$OG : GH = 1 : 2.$$

5. **Miquel's Theorem:** Let ABC be a triangle, and let D, E, and F be points on the lines BC, CA, and AB, respectively (not necessarily on the segments). Consider the three circles passing through the pairs of vertices and the corresponding points:

   $$\odot(AEF), \quad \odot(BFD), \quad \odot(CDE).$$

   Prove that these three circles have a common point (distinct from A, B, and C). This common point is called the **Miquel point** of the configuration.

6. **Menelaus' Theorem:** Let ABC be a triangle and let points D, E, F lie on the lines BC, CA, AB (possibly on their extensions). Then D, E, F are collinear if and only if

   $$\frac{BD}{DC} \cdot \frac{CE}{EA} \cdot \frac{AF}{FB} = 1.$$

7. **Ceva's Theorem (classic, segment-ratio form):** Let ABC be a triangle, and let $D \in BC$, $E \in CA$, $F \in AB$. Then lines AD, BE, CF are concurrent if and only if

$$\frac{BD}{DC} \cdot \frac{CE}{EA} \cdot \frac{AF}{FB} = 1.$$

8. **Trigonometric Ceva's Theorem:** Let ABC be a triangle, and let points D, E, and F lie on the lines BC, CA, and AB, respectively. Prove that lines AD, BE, and CF are concurrent if and only if

$$\frac{\sin \angle BAD}{\sin \angle DAC} \cdot \frac{\sin \angle CBE}{\sin \angle EBA} \cdot \frac{\sin \angle ACF}{\sin \angle FCB} = 1.$$

Problems

9. Consider an acute triangle ABC with $AB > AC$ inscribed in a circle of center O. From the midpoint D of side BC we draw line (ℓ) perpendicular to side AB that intersects it at point E. If line AO intersects line (ℓ) at point Z, prove that points A, Z, D, C are concyclic.

10. The triangle ABC has $\angle BCA = 90^\circ$. Bisector of angle $\angle CAB$ intersects the side BC in point P and bisector of angle $\angle ABC$ intersects the side AC in point Q. If M and N are projections of P and Q on side AB, find the measure of the angle $\angle MCN$.

11. Consider a cyclic quadrilateral $ABCD$ in which $BC = CD$ and $AB < AD$. Let E be a point on the side AD and F a point on the line BC such that $AE = AB = AF$. Prove that $EF \parallel BD$.

12. (JBMO 2021) Let ABC be an acute scalene triangle with circumcenter O. Let D be the foot of the altitude from A to the side BC. The lines BC and AO intersect at E. Let s be the line through E perpendicular to AO. The line s intersects AB and AC at K and L, respectively. Denote by ω the circumcircle of triangle AKL. Line AD intersects ω again at X. Prove that ω and the circumcircles of triangles ABC and DEX have a common point.

13. (JBMO 2018) Let $\triangle ABC$ and A',B',C' the symmetrics of vertex over opposite sides.The intersection of the circumcircles of $\triangle ABB'$ and $\triangle ACC'$ is A_1.B_1 and C_1 are defined similarly.Prove that lines AA_1,BB_1 and CC_1 are concurent.

14. Let I be the incenter, A_1 and B_1 midpoints of sides BC and AC of a triangle ΔABC. Denote by M and N the midpoints of the arcs AC and BC of circumcircle of ΔABC which do contain the other vertex of the triangle. If points M, I and N are collinear prove that:

$$\angle AIB_1 = \angle BIA_1 = 90^\circ$$

15. (IMO 2018) Let Γ be the circumcircle of acute triangle ABC. Points D and E are on segments AB and AC respectively such that $AD = AE$. The perpendicular bisectors of BD and CE intersect minor arcs AB and AC of Γ at points F and G respectively. Prove that lines DE and FG are either parallel or they are the same line.

16. (IGO 2025) Quadrilateral $ABCD$ is inscribed in a circle ω. Distinct points X, Y lie on the rays DB, CA such that $DA = DX, CB = CY$. Lines AX, BY intersect the side CD at points P, Q respectively. Prove that the radical axis of the two circumcircles of triangles APC, BQD and the perpendicular bisector of side AB intersect at a point on ω.

Homework

1. Choose the problem you found most interesting today and rewrite its solution in your own words.

2. Let ABC be an acute triangle inscribed in a circle of center O. If the altitudes BD, CE intersect at H and the circumcenter of $\triangle BHC$ is O_1, prove that AHO_1O is a parallelogram.

3. (EGMO 2022) Let ABC be an acute-angled triangle in which $BC < AB$ and $BC < CA$. Let point P lie on segment AB and point Q lie on segment AC such that $P \neq B$, $Q \neq C$ and $BQ = BC = CP$. Let T be the circumcenter of triangle APQ, H the orthocenter of triangle ABC, and S the point of intersection of the lines BQ and CP. Prove that T, H, and S are collinear.