1 Comments on sheaves

Purpose. This packet attempts to use not-so-scary coordinates on the projective spaces $\mathbb{P}^2$ and $\mathbb{P}^1$ to make life easier. We also supply some ideas explaining what is supposed to be happening in each of the problems in the main packet.

In particular for $\mathbb{P}^1_{\Bbbk}$ we fix the coordinate rings $\mathcal{O}_0 = \Bbbk[x]$ on $U_0 = \{[1 : x] \mid x \in \Bbbk\}$ and $\mathcal{O}_1 = \Bbbk[y]$ on $U_1 = \{[y : 1] \mid y \in \Bbbk\}$. With this notation, we have that the ring of U_{01} is $\Bbbk[x, x^{-1}] \cong \Bbbk[y, y^{-1}]$ with $x = y^{-1}$.

There is a similar story for $\mathbb{P}^2$:

$$U_0 = \{[1 : u_1 : u_2]\} \leftrightarrow \Bbbk[u_1, u_2] \tag{1}$$
$$U_1 = \{[v_0 : 1 : v_2]\} \leftrightarrow \Bbbk[v_0, v_2] \tag{2}$$
$$U_2 = \{[w_0 : w_1 : 1]\} \leftrightarrow \Bbbk[w_0, w_1] \tag{3}$$

and the functions on, for instance, U_{01} are given by the ring $\mathcal{O}_{01} = \Bbbk[u_1, u_1^{-1}, u_2] \cong \Bbbk[v_0, v_0^{-1}, v_2]$ with isomorphism given by $u_1 = v_0^{-1}$ and $u_2 = v_2$.

Recall that from global coordinates $[x_0 : x_1 : x_2]$ one recovers $u_1 = \frac{x_1}{x_0}, u_2 = \frac{x_2}{x_0}$. This is what we earlier called the restriction map res_0, and we showed that it takes homogeneous polynomials of fixed degree d in variables x_0, x_1, x_2 to polynomials of degree not more than d in u_1, u_2.

Problem Commentary to problems 2 and 3. Let us try to understand what Problem 2 says. For instance, in the case $n = 2$ we have modules

$$\mathcal{L}_0 = e_0\Bbbk[u_1, u_2],\ \mathcal{L}_1 = e_1\Bbbk[v_0, v_2],\ \mathcal{L}_2 = e_2\Bbbk[w_1, w_2],$$

and we should have three isomorphisms

$$\psi_{ij} \colon \mathcal{L}_i|_{U_{ij}} \to \mathcal{L}_j|_{U_{ij}}.$$

Now, $\mathcal{L}_0|_{U_{01}} = e_0\Bbbk[u_1^{\pm 1}, u_2]$ and we can regard $\mathcal{L}_1|_{U_{01}} = e_1\Bbbk[u_1^{\pm 1}, u_2]$ (notice that here we are hiding the isomorphism φ_{01} described above under the rug). Then the choice of isomorphism ψ_{01} is determined by the image of e_0, i.e. if you set $\psi_{01}(e_0) = g_{01}e_1$, then

$$\psi_{01}(fe_0) = f\psi_{01}(e_0) = fg_{01}e_1.$$

Now, it is important that $g_{01} \in \Bbbk[u_1^{\pm 1}, u_2]$ is invertible (otherwise, the above map is not surjective).

(a) Show that $g_{01} = \lambda u_1^m$ for some $\lambda \in \Bbbk^\times$ and $m \in \mathbb{Z}$ (this should sound surprising at first glance).

(b) Using the previous exercise, describe all sheaves on $\mathbb{P}^1$, and show also that if two sheaves only differ by choice of λ, they are naturally isomorphic. We will denote by $\mathcal{O}_{\mathbb{P}^1}(n)$ the sheaf corresponding to $g_{01} = x^n$.

(c) Show that $\psi_{12} \circ \psi_{01} = \psi_{02}$ implies $g_{12}g_{01} = g_{02} \in \Bbbk[u_1^{\pm 1}, u_2^{\pm 1}]$ (this is exactly the ring of the big open subset where $x_0, x_1, x_2 \neq 0$).

(d) Now $(s_0, s_1) \in e_0\Bbbk[x] \oplus e_1\Bbbk[y]$ is a global section of sheaf $\mathcal{L}$ on $\mathbb{P}^1$ if $\psi_{01}(s_0) = s_1$. Assume, $s_0 = f_0e_0$ and $s_1 = f_1e_1$. Show that

$$f_1 = g_{01}f_0. \tag{4}$$

Using this, describe all global sections of $\mathcal{O}_{\mathbb{P}^1}(n)$.

(e) Find a natural bijection (linear isomorphism) between the space of homogeneous polynomials of degree n in $\Bbbk[x_0, x_1]$ and the space of global sections of $\mathcal{O}_{\mathbb{P}^1}(n)$.

(f) Can you describe $\mathcal{O}_{\mathbb{P}^2}(n)$ by analogy with $\mathbb{P}^1$?

Problem Commentary to problem 4. (a) Again, let's just focus on $\mathbb{P}^1$. The sheaf $\mathcal{L}_1$ here is given by $\mathcal{L}_1^0 = e_0^1\Bbbk[x]$, $\mathcal{L}_1^1 = e_1^1\Bbbk[y]$ and $\mathcal{L}_2$ is given by $\mathcal{L}_2^0 = e_0^2\Bbbk[x]$, $\mathcal{L}_2^1 = e_1^2\Bbbk[y]$. As $\Bbbk[x]$-modules $\mathcal{L}_1^0 \cong \mathcal{L}_2^0 \cong \Bbbk[x]$. Obviously, it holds that

$$\mathcal{L}_1^0 \otimes \mathcal{L}_2^0 \cong \Bbbk[x] \otimes_{\Bbbk[x]} \Bbbk[x] \cong \Bbbk[x].$$

So, what this problem is asking is for you to define a natural map

$$\psi_{01}^{\otimes} \colon \mathcal{L}_1^0 \otimes \mathcal{L}_2^0|_{U_{01}} \cong e_0^1 \otimes e_0^2\Bbbk[x^{\pm 1}] \to e_1^1 \otimes e_1^2\Bbbk[x^{\pm 1}] \cong \mathcal{L}_1^1 \otimes \mathcal{L}_2^1|_{U_{01}},$$

based on ψ_{01}^1 and ψ_{01}^2, which are given to you (here it may be helpful again to think of them as of just some two functions g_{01}^1 and g_{01}^2 as before).

(b) For this problem just solve it in the case of $\mathbb{P}^1_{\Bbbk}$

(c) Here you are being asked to show that each sheaf over $\mathbb{P}^n$ has an inverse under taking tensor products. What is this group, by the way? Notice that the existence of this group structure is the primary reason these kinds of sheaves are called *invertible*. If you would consider modules of higher ranks, or not free ones over charts, you would get more general sheaves, but we would not need them.

2 Sheaves and divisors.

Some remarks. Here we use the notion of the field of fractions of a ring R (assume R is an integral domain, i.e., there are no zero divisors). The field $K(R)$ is defined as a collection of fractions $\frac{a}{b}$ for $a \in R, b \in R^{\times}$ up to some natural equivalence relation. For our favorite ring $\mathcal{O} = \Bbbk[u_1, \ldots, u_n]$ we have that $K(\mathcal{O})$ is the field of all fractions $\frac{P}{Q}$ with P and Q polynomials in these variables.

The word *divisor* in algebraic geometry is a replacement for the phrase "formal sum of codimension 1 subvarieties." If you want to understand surfaces in $\mathbb{P}^3$, you want to understand divisors in $\mathbb{P}^3$.

What you are about to learn is that there is a natural way to assign a divisor to a sheaf and vice versa. Here is a naive way to get a divisor D from a sheaf $\mathcal{L}$. Take some global section $s \in \Gamma(\mathcal{L})$. Then its zero-set is a collection of affine subvarieties in each chart U_i. It is not hard to see that they "glue" into a projective variety. The only problem with this strategy is that there may be no global sections. But(!) there are always rational sections.