CONDITIONAL PROBABILITY

MATH CIRCLE (ADVANCED) 2/10/2013

0) Suppose you roll a fair 6-sided die.

a) Probability you get a 6?

$1/6$

b) Suppose someone (correctly) tells you the roll was even. What would the the probability of getting a 6 be (taking this information into account)?

$1/3$

c) Suppose you roll two die. Given that the sum of the two die is ≥ 8, what is the probability that the sum is 12. Compare this with the initial (that is, given no information) probability of getting a sum of 12.

$1/15 > 1/36$

Conditional Probability: Suppose A and B are two events. The conditional probability of A given B, written $P(A|B)$, is the probability of A knowing that B happened.

Suppose Ω is a finite sample space and every outcome in Ω is equally likely. If $A, B \subseteq \Omega$ then

$$P(A|B) = \frac{|A \cap B|}{|B|} \tag{0.1}$$

In general, we have the following formula:

$$P(A|B) = \frac{P(A \cap B)}{P(B)} \tag{0.2}$$

1) Prove (0.2) assuming $A, B \subseteq \Omega$ with Ω a finite sample space with every outcome in Ω equally likely.

We use the fact that Ω is a finite sample space with every outcome equally likely as well as (0.1):

$$\frac{P(A \cap B)}{P(B)} = \frac{|A \cap B|/|\Omega|}{|B|/|\Omega|} = \frac{|A \cap B|}{|B|} = P(A|B).$$

2) Answer the following using either equation above or possibly another method:

Suppose there is an urn with 5 green, 6 red, and 4 yellow balls. Suppose you pick 4 balls without replacement.

a) What is $P(\geq 3 \text{ green balls} \mid \geq 2 \text{ green balls})$?

$$\frac{\binom{5}{3}\binom{10}{1} + \binom{5}{4}\binom{10}{0}}{\binom{5}{2}\binom{10}{2} + \binom{5}{3}\binom{10}{1} + \binom{5}{4}\binom{10}{0}}$$

b) Find the probability that you get at least 3 green balls given that the first two you pick are green. Compare this with the probability in a).

Note this is equivalent to getting ≥ 1 green balls from the last two picks:

$1 - \frac{\binom{3}{0}\binom{10}{2}}{\binom{13}{2}}$

c) Suppose you roll two 6-sided die. What the probability that the sum of the two die is 10 given that the sum is even.

$3/18$

Suppose you flip a fair coin 6 times.

d) What is the probability you get at least 3 heads, given that not all flips are tails?

$\frac{\binom{6}{3}+\binom{6}{4}+\binom{6}{5}+\binom{6}{6}}{2^6-1}$

e) What is the probability you get at least 3 heads, given that the first flip is heads? Compare this with the probability in d).

Note this is equivalent to getting ≥ 2 green balls from the last 5 flips:

$1 - \frac{\binom{5}{0}+\binom{5}{1}}{2^5}$

f) Suppose you are dealt 2 cards (without replacement) from a standard deck of cards. Given that the first card is an ace, what is the probability that the second card is an ace.

$3/51$

Product Rule: Suppose A, B are events. Then

$$P(A \cap B) = P(B \cap A) = P(B)P(A|B) = P(A)P(B|A).$$

Note this is especially useful in situations where the conditional probability is easier to calculate (as in parts b,e, and f from problem 2).

3) a) Suppose you are dealt 3 cards from a deck of cards. Find the probabilty you get three hearts.

$P(1\text{st H})P(2\text{nd H}|1\text{st H})P(3\text{rd H}|1\text{st and }2\text{nd H}) = \frac{13}{52} \cdot \frac{12}{51} \cdot \frac{11}{50}.$

b) Suppose you flip a coin 7 times. What is the probability that the first 3 flips are heads and you get exactly 4 heads total.

$P(1\text{st three are H})P(1\text{ H in last }4\text{ flips}) = \frac{1}{2^3} \cdot \frac{\binom{4}{1}}{2^4}.$

c) Suppose you roll a die and then flip a coin the number of times shown on the die. Find the probability you get 4 heads and 0 tails.

$\frac{1}{6} \cdot \frac{1}{2^4}$

d) Suppose you are dealt 2 cards from a deck of cards. Find the probability that the second card is a king.

See Law of Total Probability: $\frac{4}{52} \cdot \frac{3}{51} + \frac{48}{52} \cdot \frac{4}{51} = \frac{1}{13}$

Law of Total Probability: Suppose that $A, B_1, \dots, B_n \subseteq \Omega$, with $B_1, \dots, B_n$ pairwise disjoint (that is, B_i and B_j are disjoint for $i \neq j$) and $B_1 \cup \dots \cup B_n = \Omega$. Then

$$P(A) = \sum_{i=1}^{n} P(A|B_i).$$

4) Write out the Law of Total Probability (including the premises) for $n = 2, 3$.

5) a) Suppose you roll a die and then flip a coin the number of times shown on the die. Find the probability you get 5 heads.

You either get 5H,0T or 5H,1T: $\frac{1}{6} \cdot \frac{1}{2^5} + \frac{1}{6} \cdot \frac{\binom{6}{5}}{2^6}$.

b) Suppose you have three bags: A with 1 red and 3 green balls, B with $2R, 2G$, and C with $1R, 4G$. You roll a die and then pick a ball from bag A if you roll $1-3$, B if you roll $4-5$, and C if you roll 6. Find the probability you get a green ball.

$\frac{1}{2} \cdot \frac{3}{4} + \frac{1}{3} \cdot \frac{1}{2} + \frac{1}{6} \cdot \frac{4}{5}$

c) Suppose the setup is the same as in b, except you pick 2 balls (without replacement) from the bag. Find the probability both balls are the same color.

The balls can either be both red or both green:

$$[\frac{1}{2} \cdot 0 + \frac{1}{3} \cdot \frac{\binom{2}{2}}{\binom{4}{2}} + \frac{1}{6} \cdot 0] + [\frac{1}{2} \cdot \frac{\binom{3}{2}}{\binom{4}{2}} + \frac{1}{3} \cdot \frac{\binom{2}{2}}{\binom{4}{2}} + \frac{1}{6} \cdot \frac{\binom{4}{2}}{\binom{5}{2}}].$$

6) Prove the Law of Total Probability.

$$\begin{aligned} P(A) =& P[(A \cap B_1) \cup (A \cap B_2) \cup \dots \cup (A \cap B_n)] \\ =& P(A \cap B_1) + P(A \cap B_2) + \dots + P(A \cap B_n) \\ =& P(A|B_1)P(B_1) + \cdots P(A|B_n)P(B_n) \end{aligned}$$

7) Suppose you flip a coin n times. Suppose A_i is the event that the ith flip is heads. Calculate $P(A_i \mid \text{exactly } k \text{ of the flips were heads})$.

$$\begin{aligned} P(A_i| = k\, H's) = \frac{P(A_i \cap k\, H's)}{P(k\, H)} =& \frac{P(ith\, H)P(k-1\, H \text{ in other flips})}{P(k\, H)} \\ =& \frac{(1/2)(\binom{n-1}{k-1}/2^{n-1})}{\binom{n}{k}/2^n} = \frac{\binom{n-1}{k-1}}{\binom{n}{k}} = \frac{k}{n}. \end{aligned}$$

8) (Three Card Problem)

Suppose there are three cards. One card is red on both sides, one is green on both sides, and one has a side of each color. Suppose one card is chosen at random (i.e.

blindly out of a bag). You are able to see that one side of the card is red. What is the probability that the other side of the card is also red?

$$P(\text{both red}|\text{seen side red}) = \frac{P(\text{both red})}{P(\text{seen side red})} = \frac{1/3}{3/6} = \frac{2}{3}.$$