BAMO Training Problems

ORMC Math Circle

Problems

1. A triangle (with non-zero area) is constructed with the lengths of the sides chosen from the set

$$\{2, 3, 5, 8, 13, 21, 34, 55, 89, 144\}.$$

   Show that this triangle must be isosceles (A triangle is *isosceles* if it has at least two sides the same length.).

2. The *Fibonacci sequence* is the list of numbers that begins $1, 2, 3, 5, 8, 13$ and continues with each subsequent number being the sum of the previous two.

   Prove that when the first n elements of the Fibonacci sequence are alternately added and subtracted, the result is an element of the sequence or the negative of an element of the sequence. For example,

$$1 - 2 + 3 - 5 = -3,$$

   and 3 is an element of the Fibonacci sequence.

3. A positive integer N is called *stable* if it is possible to split the set of all positive divisors of N (including 1 and N) into two subsets that have no elements in common, which have the same sum. For example, 6 is stable, because $1 + 2 + 3 = 6$ but 10 is not stable. Is $2^{2008} \cdot 2008$ stable?

4. Suppose that for two real numbers x and y the following equality is true:

$$\left(x + \sqrt{1 + x^2}\right)\left(y + \sqrt{1 + y^2}\right) = 1.$$

   Find (with proof) the value of $x + y$.

5. Let $x_1, x_2, \ldots, x_n$ be positive numbers, with $n \geq 2$. Prove that

$$\left(x_1 + \frac{1}{x_1}\right)\left(x_2 + \frac{1}{x_2}\right) \cdots \left(x_n + \frac{1}{x_n}\right) \geq \left(x_1 + \frac{1}{x_2}\right)\left(x_2 + \frac{1}{x_3}\right) \cdots \left(x_{n-1} + \frac{1}{x_n}\right)\left(x_n + \frac{1}{x_1}\right).$$

6. Let H be the *orthocenter* of an acute triangle ABC. Draw three circles: one passing through A, B and H, another passing through B, C and H, and finally, one passing through C, A and H. Prove that the triangle whose vertices are the centers of those three circles is congruent to triangle ABC.

7. For a positive integer $n > 2$, consider the $n - 1$ fractions

$$\frac{2}{1}, \frac{3}{2}, \cdots, \frac{n}{n-1}.$$

   The product of these fractions equals n, but if you reciprocate (i.e. turn upside down) some of the fractions, the product will change. Can you make the product equal 1? Find all values of n for which this is possible and prove that you have found them all.

8. Let ABC be a triangle with D the midpoint of side AB, E the midpoint of side BC, and F the midpoint of side AC. Let k_1 be the circle passing through points A, D, and F; let k_2 be the circle passing through points B, E, and D; and let k_3 be the circle passing through points C, F, and E. Prove that circles k_1, k_2, and k_3 intersect in a point.

9. The *distinct prime factors* of an integer are its prime factors listed without repetition. For example, the distinct prime factors of 40 are 2 and 5.

   Let $A = 2^k - 2$ and $B = 2^k \cdot A$, where k is an integer ($k > 1$).

   Show that, for every choice of k,

   (a) A and B have the same set of distinct prime factors.

   (b) $A + 1$ and $B + 1$ have the same set of distinct prime factors.

10. Let k be a positive integer. Prove that there exist integers x and y, neither of which is divisible by 3, such that $x^2 + 2y^2 = 3^k$.

11. (Romania TST 2019) Let be three positive integers a, b, c and a function $f : \mathbb{N} \longrightarrow \mathbb{N}$ defined as

$$f(n) = \begin{cases} n - a, & n > c \\ f\left(f(n + b)\right), & n \leq c \end{cases}.$$

    Determine the number of fixed points this function has.

12. (EGMO 2021) The number 2021 is fantabulous. For any positive integer m, if any element of the set $\{m, 2m + 1, 3m\}$ is fantabulous, then all the elements are fantabulous. Does it follow that the number 2021^{2021} is fantabulous?

Homework

1. Choose the problem you found most interesting today and rewrite its solution in your own words.

2. Find all natural numbers n such that when we multiply all divisors of n, we will obtain 10^9. Prove that your number(s) n works and that there are no other such numbers.

3. Since $24 = 3 + 5 + 7 + 9$, the number 24 can be written as the sum of at least two consecutive odd positive integers.

   (a) Can 2005 be written as the sum of at least two consecutive odd positive integers? If yes, give an example of how it can be done. If no, provide a proof why not.

   (b) Can 2006 be written as the sum of at least two consecutive odd positive integers? If yes, give an example of how it can be done. If no, provide a proof why not.

4. In an acute triangle ABC let K, L, and M be the midpoints of sides AB, BC, and CA, respectively. From each of K, L, and M drop two perpendiculars to the other two sides of the triangle; e.g., drop perpendiculars from K to sides BC and CA, etc. The resulting 6 perpendiculars intersect at points Q, S, and T as in the figure to form a hexagon $KQLSMT$ inside triangle ABC. Prove that the area of the hexagon $KQLSMT$ is half of the area of the original triangle ABC.