

CONDITIONAL PROBABILITY

MATH CIRCLE (ADVANCED) 2/10/2013

0) Suppose you roll a fair 6-sided die.

a) Probability you get a 6?

b) Suppose someone (correctly) tells you the roll was even. What would the the probability of getting a 6 be (taking this information into account)?

c) Suppose you roll two die. Given that the sum of the two die is ≥ 8 , what is the probability that the sum is 12. Compare this with the initial (that is, given no information) probability of getting a sum of 12.

Conditional Probability: Suppose A and B are two events. The conditional probability of A given B , written $P(A|B)$, is the probability of A knowing that B happened. Suppose Ω is a finite sample space and every outcome in Ω is equally likely. If $A, B \subseteq \Omega$ then

$$(0.1) \quad P(A|B) = \frac{|A \cap B|}{|B|}$$

In general, we have the following formula:

$$(0.2) \quad P(A|B) = \frac{P(A \cap B)}{P(B)}$$

1) Prove (0.2) assuming $A, B \subseteq \Omega$ with Ω a finite sample space with every outcome in Ω equally likely.

2) Answer the following using either equation above or possibly another method:

Suppose there is an urn with 5 green, 6 red, and 4 yellow balls. Suppose you pick 4 balls without replacement.

a) What is $P(\geq 3 \text{ green balls} \mid \geq 2 \text{ green balls})$?

b) Find the probability that you get at least 3 green balls given that the first two you pick are green. Compare this with the probability in a).

c) Suppose you roll two 6-sided die. What the probability that the sum of the two die is 10 given that the sum is even.

Suppose you flip a fair coin 6 times.

d) What is the probability you get at least 3 heads, given that not all flips are tails?

e) What is the probability you get at least 3 heads, given that the first flip is heads? Compare this with the probability in d).

f) Suppose you are dealt 2 cards (without replacement) from a standard deck of cards. Given that the first card is an ace, what is the probability that the second card is an ace.

Product Rule: Suppose A, B are events. Then

$$P(A \cap B) = P(B \cap A) = P(B)P(A|B) = P(A)P(B|A).$$

Note this is especially useful in situations where the conditional probability is easier to calculate (as in parts b,e, and f from problem 2).

- 3) a) Suppose you are dealt 3 cards from a deck of cards. Find the probability you get three hearts.
- b) Suppose you flip a coin 7 times. What is the probability that the first 3 flips are heads and you get exactly 4 heads total.
- c) Suppose you roll a die and then flip a coin the number of times shown on the die. Find the probability you get 4 heads and 0 tails.
- d) Suppose you are dealt 2 cards from a deck of cards. Find the probability that the second card is a king.

Law of Total Probability: Suppose that $A, B_1, \dots, B_n \subseteq \Omega$, with B_1, \dots, B_n pairwise disjoint (that is, B_i and B_j are disjoint for $i \neq j$) and $B_1 \cup \dots \cup B_n = \Omega$. Then

$$P(A) = \sum_{i=1}^n P(A|B_i).$$

4) Write out the Law of Total Probability (including the premises) for $n = 2, 3$.

5) a) Suppose you roll a die and then flip a coin the number of times shown on the die. Find the probability you get 5 heads.

b) Suppose you have three bags: A with 1 red and 3 green balls, B with 2R, 2G, and C with 1R, 4G. You roll a die and then pick a ball from bag A if you roll 1 – 3, B if you roll 4 – 5, and C if you roll 6. Find the probability you get a green ball.

c) Suppose the setup is the same as in b, except you pick 2 balls (without replacement) from the bag. Find the probability both balls are the same color.

6) Prove the Law of Total Probability.

7) Suppose you flip a coin n times. Suppose A_i is the event that the i th flip is heads. Calculate $P(A_i \mid \text{exactly } k \text{ of the flips were heads})$.

8) (Three Card Problem)

Suppose there are three cards. One card is red on both sides, one is green on both sides, and one has a side of each color. Suppose one card is chosen at random (i.e. blindly out of a bag). You are able to see that one side of the card is red. What is the probability that the other side of the card is also red?