The mathematics of billiards

1. Simple billiards

The basic setup of this packet is the familiar billiards game.

1.1 Definition. Suppose that we have a polygon P. Inside it a poinlike ball is moving in a straight line, bouncing off the walls with the reflection (equal angle) law. We define the path of its motion to be a *billiards trajectory*.

1.2 Scholium. If the billiards ball hits a corner, we declare the behavior after that to be *undefined*.

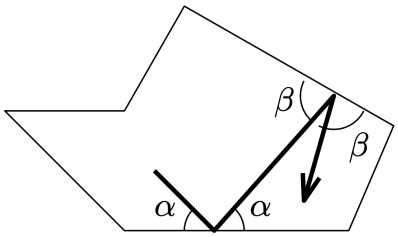


Figure 1: A billiards trajectory.

1.3 Definition. A *rational angles polygon* is a polygon P whose interior angles are all rational multiples of 360° (e.g. 44° but not $(22/\pi)^\circ$).

1.4 Exercise. Give an example of a rational angles polygon P and a polygon P which is not rational angles. Show that if a rational angles polygon P has interior angles which are all multiples of $180^\circ/M$ then any single billiards trajectory travels in at most $2M$ different directions. Prove that this is in general sharp.

2. Periodic trajectories

Whither the periodic trajectories?

Unattributed

We shall not cease from exploration
And the end of all our exploring
Will be to arrive where we started
And know the place for the first time.

T.S. Eliot, *Little Gidding*

Keep Ithaka always in your mind.
Arriving there is what you're destined for.

Constantine Cavafy, *Ithaka*

Round the world! There is much in that sound to inspire proud feelings; but whereto does all that circumnavigation conduct? Only through numberless perils to the very point whence we started, where those that we left behind secure, were all the time before us.

Were this world an endless plain, and by sailing eastward we could for ever reach new distances, and discover sights more sweet and strange than any Cyclades or Islands of King Solomon, then there were promise in the voyage. But in pursuit of those far mysteries we dream of, or in tormented chase of that demon phantom that, some time or other, swims before all human hearts; while chasing such over this round globe, they either lead us on in barren mazes or midway leave us whelmed.

Herman Melville, commenting on the dynamical consequences of the compactness of the topological 2-sphere.

2.1 Definition. A periodic trajectory is a billiards trajectory which returns to the point from where it started. The period of a trajectory is the number of edges it hits before it returns to its starting place.

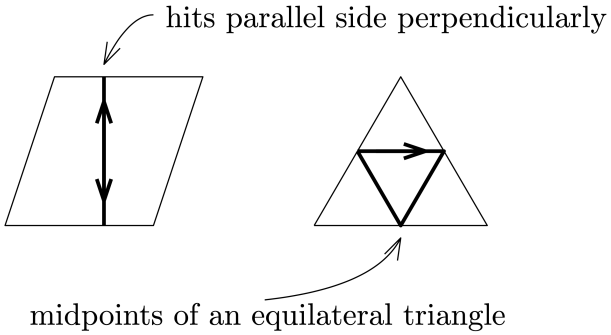


Figure 2: Two examples of a periodic trajectory.

2.1. A suprising open problem

Problem 1 (Open problem)**.** Does every triangle have a periodic billiards trajectory?

2.2 Exercise. Let P be an acute triangle (all angles $< 90°$). Draw a periodic billiards trajectory in P with period 3.

2.3 Exercise. For billiards in an equilateral triangle, find a period trajectory with more than 6 bounces. Draw a sketch to show why this construction works.

2.4 Exercise. For billiards in a 30/60/90 triangle, find a periodic trajectory other than the 6 bounce trajectory that exists in any right angled triangle.

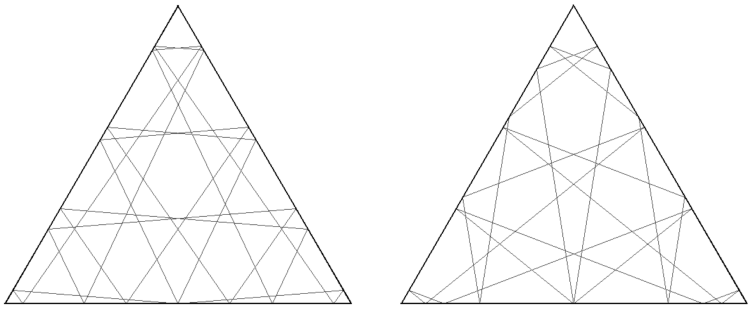
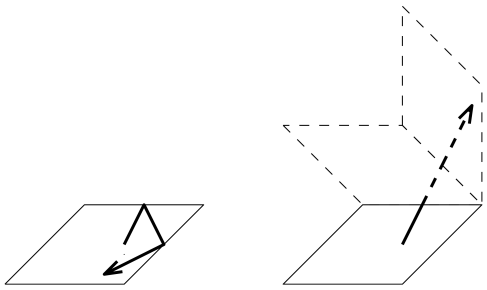


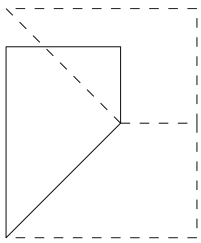
Figure 3: Periodic orbits on a triangle, what is their period?

3. Tiling by reflection

You may have discovered in the previous exercises that there is a useful trick. If we think of the walls of a polygon as mirrors, then we can discover periodic trajectories more easily.



3.1 Scholium. This doesn't work with every polygon...



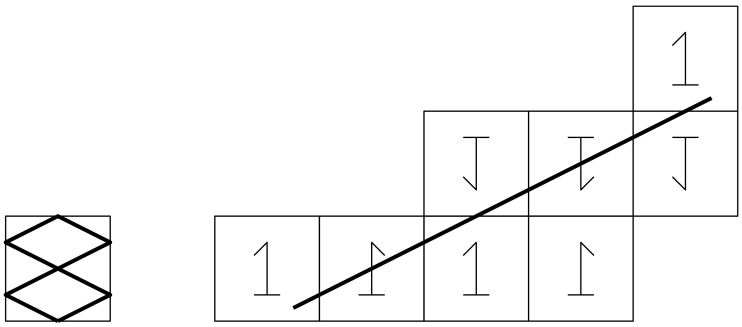
3.2 Example. Here is an example of how tiling by reflection can allow you to find a 6-periodic trajectory for billiards in a square.

3.3 Exercise. For billiards in a square, find a 14-bounce periodic trajectory.

3.4 Exercise. Can you state clearly how tiling by reflection allows you to find a periodic trajectory? How do you unfold the polygon? Are there any conditions on the first and last polygon?

3.5 Exercise. Use the tiling by reflection trick to demonstrate a 6-periodic trajectory in an equilateral triangle.

3.6 Exercise. Find two different five bounce trajectories in a regular pentagon.



3.7 Exercise. In a right angle triangle, find a 6-periodic trajectory which hits the walls in only 4 distinct places.

3.8 Exercise. If we have a periodic trajectory with an odd number N of bounces, how can we modify this trajectory to find one with $2N$ bounces?

3.1. Orbit types

3.9 Definition. For a periodic trajectory, its *orbit type* is the list of edges it hits.

3.10 Exercise. We play billiards in a square. For a billiards trajectory (not necessarily periodic) we write "h" whenever we hit a horizontal wall and "v" whenever we hit a vertical wall. Looking at the first few bounces, which of the following are possible:

1. $hhhhh$,
2. $vvhhh$,
3. $vvhhv$.

3.11 Exercise. Playing billiards in a square, we follow a trajectory which is not purely vertical or horizontal, let h_n be the number of times the ball hits the horizontal wall in the first n bounces and similarly define v_n. What can you say about

- the sum $h_n + v_n$?
- the ratio $\frac{h_n}{v_n}$ when n is very large?

Problem 2. For billiards in a 30/60/90 triangle, how can we construct a 16-bounce periodic trajectory which hits the sides in the following order (l =

longest, m = medium, s = shortest): $mlmlmlslmlsmlmls$?

3.12 Exercise. Take a thin wedge which lies between the line segments

$$l_1 = \{x \geq 0, y = 0\}, \quad l_2 = \{x \geq 0, y = 10^{-6}x\}$$

for some small positive number a. Shoot a billiards ball into the wedge, going left in a horizontal directions. The trajectory will bounce a number of times before it emerges and goes off to infinity with no more bounces. How many bounces will there be?

4. The notion of a "phase space"

> What is "phase"? This vague word can mean many things, but usually "phase" connotes state. For instance: we might say the three phases of matter are `solid, liquid` and `gas`. In this example, we mean that matter has three states: solid, liquid and gas. If a magnet ceases to work after being heated we might say it has undergone a "phase transition", i.e. its "phase" has changed from `working` to `not working`. If an ice cube melts into water it also said to have undergone a "phase transition".

More concretely, *phase space* typically refers to a set Ω which parametrizes all the information about a system.

4.1 Definition. The phase (or state) of a billiards ball is its position and its direction of motion.

4.2 Definition (The billiards phase space)**.** Let P be a billiards table with ∂P its boundary. The phase space for billiards is

$$\Omega \coloneqq \partial P \times (0, \pi)$$

which records the position on the boundary and the angle of motion.

4.3 Definition (The billiards map on the phase space)**.** The billiards map is defined on the phase space is defined as follows, for a point $(p, \theta) \in \Omega \coloneqq \partial P \times (0, \pi)$ start a billiards ball at p moving at unit speed in the direction θ, let the ball run until it hits ∂P again at a new point $p' \in \partial P$, reflect the angle of the ball along the tangent to ∂P at p' to obtain a new angle $\theta' \in (0, \pi)$, this defines a map

$$T : \Omega \to \Omega.$$

4.4 Exercise (Comprehension)**.** Is the map T well defined? Think about a polygonal shape?

4.5 Exercise (Comprehension)**.** Draw a polygonal shape with a billiards trajectory. Indicate on your drawing the points in phase space and the action of the billiards map.

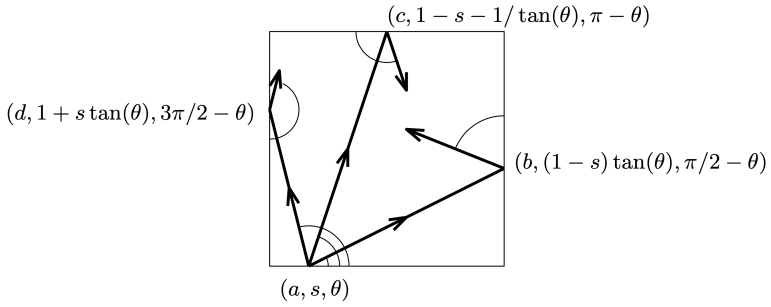
4.1. Conservation of areas

Let P be a polygon and T the billiards map. Let's write

$$T(e, s, \theta) = (e', s', \theta')$$

where e is an edge, s is the distance from the vertex in the clockwise direction, and θ is an angle.

4.6 Example. Here is an example of the billiards map written in coordinates. The sides of the square are labeled a, b, c, d. We see how three different choices of θ give us three different alternatives for the billiards map.



4.7 Exercise. By reasoning about

$$T(e, s + \Delta s, \theta)$$

and

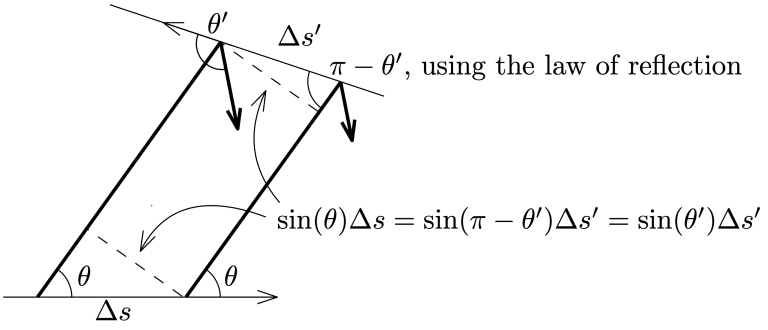
$$T(e, s, \theta + \Delta\theta)$$

can you compute

$$\frac{\partial s'}{\partial s}, \frac{\partial \theta'}{\partial s}, \text{ and } \frac{\partial \theta'}{\partial \theta}?$$

Don't worry about $\frac{\partial s'}{\partial \theta}$.

Here is a diagram that helps to think about $\partial s'/\partial s$:



4.8 Exercise. We define the "total derivative" of T at (e, s, θ) as

$$\nabla T(e, s, \theta) = \begin{pmatrix} \frac{\partial s'}{\partial s} & \frac{\partial s'}{\partial t} \\ \frac{\partial t'}{\partial s} & \frac{\partial t'}{\partial t} \end{pmatrix}.$$

The matrix $\nabla T(e, s, \theta)$ describes the "infinitesimal" amount of stretching that T applies to the phase space Ω.

For every (e, s, θ), what is the determinant of $\nabla T(e, s, \theta)$? Recall the definition of the determinant for a 2 by 2 matrix

$$\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} \coloneqq ad - bc.$$

4.9 Exercise. If S is the unit square in the plane, and

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

then what is the area of $A(S)$? How does the determinant tell you about the change of areas?

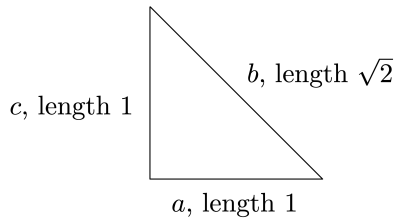
4.10 Theorem (Liouville Theorem)**.** *In (e, s, θ) coordinates, the map T is area preserving.*

4.11 Exercise. Let $U \subset \Omega$ be any subset with positive area. Then, there is some $p \in U$ and an $M > 0$ so that

$$T^M(p) = T(T(\cdots(p))\cdots) \in U.$$

Hint: area preservation is crucial, it is also crucial to think about the map T^{-1} (coming from billiards running in reverse).

4.12 Exercise. Write down explicitly, in (s, θ)-coordinates on phase space, the billiards transformation T for the right-angled triangle drawn below, in the case where the starting point lies on the edge a.



Problem 3. Take any polygon. Choose an initial boundary point and inwards pointing direction randomly. For each edge of our polygon, what is the probability that the 100th bounce will occur on that edge.

Problem 4 (Outer billiards)**.** Similar to Billiards there is a game called outer billiards with the following rules:

> Take a polygon P start at a point x_0 *outside the polygon.* Among all the vertices of the polygon in our field of view, choose the one on the furthest right and call it v_0 (if there is more than 1 then this procedure is not well defined). Travel from x_0 to v_0 then continue straight the same length, which brings us to a point $x_1 = x_0 + 2(v_0 - x_0) = 2v_0 - x_0$. This is our next point x_1. Restarting the procedure at x_1 we can obtain the outer billiards trajectory.

For outer billiards, there is an analogue of the Liouville's theorem which is much simpler (not involving any kind of transition to phase space). What is it?

5. Billiards on curved domains

The basic rules for curved billiards are given here below.

The shortest path principle Billard balls move according to the shortest path principle.

5.1 Exercise. In Figure 5, the length of the path is

$$S(t) = \|c(t)-p\|+\|c(t)-q\| = \sqrt{(c(t)-p)\cdot(c(t)-p)}+\sqrt{(c(t)-q)\cdot(c(t)-q)}.$$

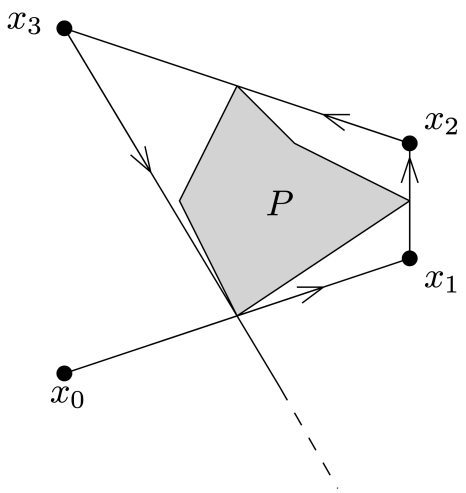


Figure 4: The outer billiards trajectory.

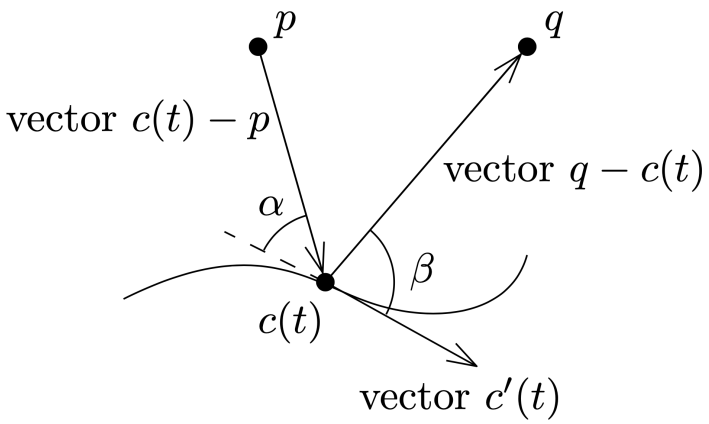


Figure 5: Understanding the shortest path principle.

In the previous line we use the notation

$$v \cdot w = v_1 w_1 + v_2 w_2$$

where $v, w \in \mathbb{R}^2$ are vectors.

Prove that

$$S'(t) = \frac{(c(t) - p) \cdot c'(t)}{\|c(t) - p\|} + \frac{(c(t) - q) \cdot}{\|c(t) - q\|}.$$

Hint: you need a little bit of differentiation, and a little bit of courage.

5.2 Exercise. Prove that $S'(t) = 0$ if and only if the angles are equal.

5.3 Definition. A billiards ball bounces off curved walls according to either:

- the shortest path principle,
- the equal angles principle.

5.4 Exercise. Let $p, q \in \mathbb{R}^2$, for a number $s > |p - q|$ we define an ellipse as

$$\mathcal{E} = \mathcal{E}(p, q, s) \coloneqq \{v \in \mathbb{R}^2 : \|v - p\| + \|v - q\| = s\}.$$

The points p, q are called the foci of $\mathcal{E}$. Prove that if a billiards ball starts at p, it will pass through q after bouncing off the wall.

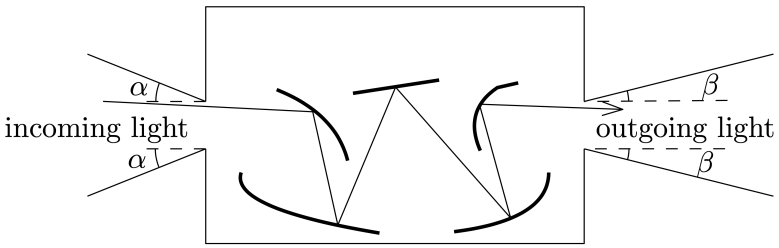
5.5 Exercise. A parabola has a single focus point. For concreteness, let's look at $y = x^2$ where the focus point is $(x, y) = (0, 1/4)$. Show that every ray of light hitting the parabola vertically from above is reflected so that it goes through the focus point.

What is an application of this to acoustics?

5.6 Exercise. Draw all possible 7 bounce periodic trajectories on the circle. You don't have to prove that you have all of them.

5.1. Squeezing light

Consider the following setup: we have a box with openings on either side and mirrors inside. The light enters through an opening of width l and angle α and leaves through an opening of width m and angle β



5.7 Exercise. Prove that if all the light which enters the box also leaves the box and if $l = m$, then $\alpha \leq \beta$.

Prove that if all the light which enters the box also leaves the box and if $\alpha = \beta$ then $l \leq m$.

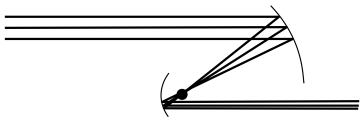
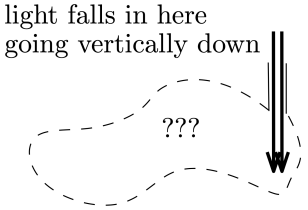
Hint: prove the following problem.

5.8 Exercise. Prove more generally for the optical contraption to be possible it must be that

$$\frac{l}{m} \leq \frac{\sin(\beta)}{\sin(\alpha)}.$$

Hint: think about the area preservation property of billiards.

5.9 Exercise. A reflecting telecope uses two parabolae with the same focus and has the following effect: light coming in through one hole (exactly) horizontally will leave, again (exactly) horizontally, through the second hold; but the second hole is smaller than the first. Why does it not contradict area preservation in phase space?



5.10 Exercise. One can make the following two-dimensional contraption with a hole which has the following property: all light falling into the hole (exactly) horizontally remains in the contraption forever. How?