



Oleg Gleizer  
prof1140g@math.ucla.edu

Brendan Connelly  
brendanconnelly@ucla.edu

## Geometry - Introduction to Linear Algebra Packet 2

### Preliminary - Warm Up

**Problem 1.** When Mr. Smith cashed a check for  $x$  dollars and  $y$  cents, he received instead  $y$  dollars and  $x$  cents, and found that he had two cents more than twice the proper amount. For how much was the check written?

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## 1. Vector spaces and subspaces recap

**Note 1.** Last time: we introduced *vector spaces* (where we can add vectors and scale them by real numbers) and *subspaces* (subsets closed under addition and scalar multiplication).

This packet focuses on a new idea: **how to describe vectors efficiently**. In geometry, physics, economics, and finance, we often have a few “basic moves” and want to build everything from them. That idea is called span and leads to the notions of linear independence and basis. Then we will study linear transformations: rules that respect addition and scaling (these model many real-world processes).

**Problem 2** (Recap: subspace test). Let  $V = \mathbb{R}^3$ . Consider the subsets:

$$W_1 = \{(x, y, z) \in \mathbb{R}^3 \mid x + y + z = 0\}, \quad W_2 = \{(x, y, z) \in \mathbb{R}^3 \mid x + y + z = 1\}.$$

1. Which of  $W_1$  or  $W_2$  is a subspace of  $\mathbb{R}^3$ ? Explain.
2. For the one that *is* a subspace, give one nonzero vector in it.

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## 2. Linear combinations and span

**Note 2.** A linear combination means “add scaled copies.” This is the core operation in linear algebra.

**Definition 1.** Let  $V$  be a vector space and let  $\vec{v}_1, \dots, \vec{v}_k \in V$ . A linear combination of  $\vec{v}_1, \dots, \vec{v}_k$  is a vector of the form

$$a_1\vec{v}_1 + \dots + a_k\vec{v}_k,$$

where  $a_1, \dots, a_k \in \mathbb{R}$ .

**Definition 2.** Let  $V$  be a vector space and let  $S \subseteq V$ . The span of  $S$ , written  $\text{span}(S)$ , is the set of all linear combinations of finitely many vectors from  $S$ . Equivalently,

$$\text{span}(S) = \{ a_1\vec{v}_1 + \dots + a_k\vec{v}_k \mid k \in \mathbb{N}, \vec{v}_i \in S, a_i \in \mathbb{R} \}.$$

**Note 3.** Geometric meaning: if your “allowed moves” are vectors in  $S$ , then  $\text{span}(S)$  is the set of all moves you can achieve by combining them.

**Example 1** (Physics / controls: two thrusters in the plane). A small robot can fire two thrusters. Thruster 1 produces displacement  $\vec{x}_1 = (2, 1)$  and thruster 2 produces displacement  $\vec{x}_2 = (1, 2)$  (in meters, say). If we fire thruster 1 for time  $a$  and thruster 2 for time  $b$ , the net displacement is

$$a\vec{x}_1 + b\vec{x}_2.$$

Question: can the robot reach  $\vec{v} = (5, 4)$ ? That means solving

$$a(2, 1) + b(1, 2) = (5, 4).$$

In coordinates this is the system  $2a + b = 5$  and  $a + 2b = 4$ , which has the unique solution  $a = 2$ ,  $b = 1$ . So  $\vec{v} \in \text{span}\{\vec{x}_1, \vec{x}_2\}$ .

**Problem 3** (Basic span computations). Let  $\vec{v}_1 = (1, 0)$ ,  $\vec{v}_2 = (0, 1)$ ,  $\vec{v}_3 = (1, 1)$  in  $\mathbb{R}^2$ .

1. Describe  $\text{span}\{\vec{v}_1, \vec{v}_2\}$ . (Say it in words.)
2. Describe  $\text{span}\{\vec{v}_3\}$ . (Draw a picture: what set of points is it?)
3. Is  $(3, -2) \in \text{span}\{\vec{v}_3\}$ ? Why or why not?

**Problem 4** (Geometry: spanning a line through the origin). Let  $\vec{u} = (3, -6) \in \mathbb{R}^2$ . Consider the set  $L = \text{span}\{\vec{u}\}$ .

1. Show that  $L = \{(x, y) \in \mathbb{R}^2 \mid y = -2x\}$ .
2. Explain geometrically why  $\text{span}\{\vec{u}\}$  must be a line through the origin.

**Example 2** (Economics / finance (simple): exposure vectors). Imagine a simplified world with two “risk factors”: Tech and Energy. A portfolio has an *exposure vector*

$$(\text{Tech exposure, Energy exposure}) \in \mathbb{R}^2.$$

A trade changes exposure by adding a vector. For example, a trade might change exposure by  $\vec{t}_1 = (1, 0)$  (more Tech), or  $\vec{t}_2 = (0, 1)$  (more Energy), or  $\vec{t}_3 = (1, 1)$  (more of both).

If a trader can perform trades in a set  $S$ , then the set of all exposure changes they can produce is  $\text{span}(S)$ .

**Problem 5** (Finance interpretation: reachable exposures). Suppose available trades are  $\vec{x}_1 = (2, 1)$  and  $\vec{x}_2 = (1, 2)$ .

1. Can you create an exposure change of  $(3, 3)$ ? If yes, find coefficients  $a, b$  such that  $a\vec{x}_1 + b\vec{x}_2 = (3, 3)$ .
2. Can you create an exposure change of  $(1, 0)$ ? If yes, find coefficients; if not, explain.
3. Explain in words what it would mean if  $\text{span}\{\vec{x}_1, \vec{x}_2\} = \mathbb{R}^2$ .

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### 3. Linear independence: removing redundancy

**Note 4.** Sometimes you have a list of “basic moves” but one is unnecessary because it can be built from the others. That idea is linear dependence. Independence means no redundancy.

**Definition 3.** Vectors  $\vec{v}_1, \dots, \vec{v}_k \in V$  are linearly independent if the only way to get the zero vector from them is the trivial combination:

$$a_1\vec{v}_1 + \dots + a_k\vec{v}_k = \vec{0} \quad \Rightarrow \quad a_1 = \dots = a_k = 0.$$

If there is a nontrivial solution (not all coefficients zero), the vectors are linearly dependent.

**Note 5.** Interpretation:  $\vec{v}_1, \dots, \vec{v}_k$  are independent if no vector in the list can be made from the others.

**Theorem 1** (Dependence means redundancy). If  $\vec{v}_1, \dots, \vec{v}_k$  are linearly dependent, then one of the vectors can be written as a linear combination of the others.

**Problem 6** (Prove the redundancy theorem). Assume  $\vec{v}_1, \dots, \vec{v}_k$  are dependent. Then there exist  $a_1, \dots, a_k$ , not all zero, such that

$$a_1\vec{v}_1 + \dots + a_k\vec{v}_k = \vec{0}.$$

1. Choose an index  $j$  such that  $a_j \neq 0$ .
2. Solve for  $\vec{v}_j$  in terms of the others.

**Problem 7** (Redundant trades). Let  $\vec{x}_1 = (1, 0)$ ,  $\vec{x}_2 = (0, 1)$ ,  $\vec{x}_3 = (1, 1)$  in  $\mathbb{R}^2$ .

1. Show that  $\{\vec{x}_1, \vec{x}_2, \vec{x}_3\}$  is linearly dependent by finding a nontrivial combination that equals  $\vec{0}$ .
2. Which trade is redundant? (Write it as a combination of the other two.)
3. Explain in finance language: what does “redundant trade” mean here?

**Problem 8** (Geometry: independent vs dependent in  $\mathbb{R}^2$ ). Let  $\vec{u} = (2, 1)$  and  $\vec{v} = (4, 2)$ .

1. Are  $\vec{u}$  and  $\vec{v}$  linearly independent? Why?
2. Describe  $\text{span}\{\vec{u}, \vec{v}\}$ . Is it a line or all of  $\mathbb{R}^2$ ?

**Problem 9** (A useful skill: quick independence test in  $\mathbb{R}^2$ ). Let  $\vec{u} = (a, b)$  and  $\vec{v} = (c, d)$  in  $\mathbb{R}^2$ .

1. Show that  $\vec{u}$  and  $\vec{v}$  are linearly dependent exactly when one is a scalar multiple of the other.
2. Use your result to decide whether  $(3, 5)$  and  $(6, 10)$  are independent.

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## 4. Bases and coordinates: choosing a measuring system

**Note 6.** Coordinates are not part of a vector. Coordinates depend on a choice of basis. A basis is a list of vectors that (i) can build every vector (spanning) and (ii) is not redundant (independent).

**Definition 4.** A list of vectors  $B = (\vec{b}_1, \dots, \vec{b}_n)$  in a vector space  $V$  is a basis of  $V$  if

1.  $V = \text{span}\{\vec{b}_1, \dots, \vec{b}_n\}$  (spans  $V$ ),
2.  $\vec{b}_1, \dots, \vec{b}_n$  are linearly independent.

**Definition 5.** If  $B = (\vec{b}_1, \dots, \vec{b}_n)$  is a basis of  $V$ , then every  $\vec{v} \in V$  can be written uniquely as

$$\vec{v} = a_1\vec{b}_1 + \dots + a_n\vec{b}_n.$$

The scalars  $a_1, \dots, a_n$  are called the coordinates of  $\vec{v}$  in the basis  $B$ , written

$$[\vec{v}]_B = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}.$$

**Theorem 2** (Coordinates are unique). If  $B$  is a basis, then each vector  $\vec{v} \in V$  has exactly one coordinate vector  $[\vec{v}]_B$ .

**Problem 10** (Prove uniqueness of coordinates). Let  $B = (\vec{b}_1, \dots, \vec{b}_n)$  be a basis. Suppose

$$\vec{v} = a_1\vec{b}_1 + \dots + a_n\vec{b}_n \quad \text{and} \quad \vec{v} = c_1\vec{b}_1 + \dots + c_n\vec{b}_n.$$

1. Subtract the two equations to get a linear combination that equals  $\vec{0}$ .
2. Use linear independence to conclude  $a_i = c_i$  for all  $i$ .

**Example 3** (Geometry: a skew coordinate system (city streets)). In a city, streets run in two directions given by vectors

$$\vec{b}_1 = (2, 0), \quad \vec{b}_2 = (1, 1).$$

A location  $\vec{v} = (5, 2)$  (in standard coordinates) can also be described by how many blocks you walk along  $\vec{b}_1$  and along  $\vec{b}_2$ . That means finding  $a, b$  such that

$$\vec{v} = a\vec{b}_1 + b\vec{b}_2.$$

This is a coordinate problem:  $[\vec{v}]_B = \begin{pmatrix} a \\ b \end{pmatrix}$  where  $B = (\vec{b}_1, \vec{b}_2)$ .

**Problem 11** (Compute coordinates in a skew basis). Let  $B = (\vec{b}_1, \vec{b}_2)$  with  $\vec{b}_1 = (2, 0)$  and  $\vec{b}_2 = (1, 1)$ .

1. Find  $[\vec{v}]_B$  for  $\vec{v} = (5, 2)$ .
2. Find  $[\vec{w}]_B$  for  $\vec{w} = (1, 3)$ .
3. Find a vector  $\vec{u} \neq \vec{0}$  such that  $[\vec{u}]_B = \begin{pmatrix} 0 \\ 2 \end{pmatrix}$ . (Then write  $\vec{u}$  in standard coordinates.)

**Example 4** (Algebra meets geometry: polynomial coordinates). Let  $P_2(\mathbb{R})$  be polynomials of degree  $\leq 2$ . A natural basis is

$$B = (1, x, x^2).$$

Then  $p(x) = 7 - 3x + 2x^2$  has coordinates  $[p]_B = \begin{pmatrix} 7 \\ -3 \\ 2 \end{pmatrix}$ .

But we can choose different bases, for example

$$C = (1, (x-1), (x-1)^2),$$

which is useful when expanding around  $x = 1$ .

**Problem 12** (Coordinates in a nonstandard polynomial basis). Let  $C = (1, (x-1), (x-1)^2)$  in  $P_2(\mathbb{R})$ .

1. Write  $p(x) = x^2$  in the form  $p(x) = a \cdot 1 + b(x-1) + c(x-1)^2$ .
2. Conclude  $[x^2]_C = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$ .
3. Explain (in words) why changing bases can be useful, even though the vector  $p$  itself does not change.

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## 5. Dot products and inner products: measuring length and angles

**Note 7.** So far, we know how to *add* vectors and *scale* them. To do geometry with vectors, we also want a way to *measure* things: length, angles, and how similar two vectors are. The dot product is the standard tool in  $\mathbb{R}^n$ , and it is a special case of a more general idea called an *inner product*.

**Definition 6** (Inner product). *Let  $V$  be a real vector space. An inner product on  $V$  is a function*

$$\langle \cdot, \cdot \rangle : V \times V \rightarrow \mathbb{R}$$

*that assigns a real number to each pair of vectors and satisfies, for all  $\vec{u}, \vec{v}, \vec{w} \in V$  and  $c \in \mathbb{R}$ :*

1. **Linearity in the first input:**  $\langle \vec{u} + \vec{v}, \vec{w} \rangle = \langle \vec{u}, \vec{w} \rangle + \langle \vec{v}, \vec{w} \rangle$  and  $\langle c\vec{u}, \vec{w} \rangle = c\langle \vec{u}, \vec{w} \rangle$ .
2. **Symmetry:**  $\langle \vec{u}, \vec{v} \rangle = \langle \vec{v}, \vec{u} \rangle$ .
3. **Positive definiteness:**  $\langle \vec{v}, \vec{v} \rangle \geq 0$ , and  $\langle \vec{v}, \vec{v} \rangle = 0$  only when  $\vec{v} = \vec{0}$ .

**Problem 13** (Linearity in the second variable). In a real inner product, (1) and (2) imply linearity in the second variable. State this in an analogous way as (1) and prove it.

**Definition 7** (Dot product in  $\mathbb{R}^n$ ). For vectors  $\vec{x} = (x_1, \dots, x_n)$  and  $\vec{y} = (y_1, \dots, y_n)$  in  $\mathbb{R}^n$ , the dot product is

$$\vec{x} \cdot \vec{y} := x_1 y_1 + \dots + x_n y_n.$$

This is an inner product on  $\mathbb{R}^n$  (with  $\langle \vec{x}, \vec{y} \rangle = \vec{x} \cdot \vec{y}$ ).

**Definition 8** (Norm (length) from an inner product). If  $\langle \cdot, \cdot \rangle$  is an inner product on  $V$ , the norm of  $\vec{v} \in V$  is

$$\|\vec{v}\| \stackrel{\text{def}}{=} \sqrt{\langle \vec{v}, \vec{v} \rangle}.$$

In  $\mathbb{R}^n$  with the dot product, this gives  $\|\vec{v}\| = \sqrt{v_1^2 + \dots + v_n^2}$ .

**Theorem 3** (Angle formula in  $\mathbb{R}^n$ ). If  $\vec{u}, \vec{v} \in \mathbb{R}^n$  are nonzero, then there is an angle  $\theta \in [0, \pi]$  between them satisfying

$$\vec{u} \cdot \vec{v} = \|\vec{u}\| \|\vec{v}\| \cos \theta.$$

In particular,  $\vec{u} \cdot \vec{v} = 0$  exactly when the vectors are perpendicular.

**Note 8.** Interpretation:  $\vec{u} \cdot \vec{v}$  is large and positive when  $\vec{u}$  points in a similar direction to  $\vec{v}$ , large and negative when they point in opposite directions, and 0 when they are orthogonal (perpendicular).

**Problem 14** (Compute dot products and norms). Let  $\vec{u} = (2, -1, 3)$  and  $\vec{v} = (1, 4, -2)$ .

1. Compute  $\vec{u} \cdot \vec{v}$ .
2. Compute  $\|\vec{u}\|$  and  $\|\vec{v}\|$ .
3. Are  $\vec{u}$  and  $\vec{v}$  perpendicular? Explain using your dot product.

**Problem 15** (A dot product identity). Let  $\vec{u}, \vec{v} \in \mathbb{R}^n$ . Expand both sides and verify the identity

$$\|\vec{u} + \vec{v}\|^2 = \|\vec{u}\|^2 + 2(\vec{u} \cdot \vec{v}) + \|\vec{v}\|^2.$$