1 Projective algebraic geometry

Problem 0. We suggest that you remind yourselves of the notions that we learned about two weeks ago: rings (R), ideals ($I \subset R$), quotient rings (R/I), zero-sets ($V(I)$), and ring localizations (R_f for an element $f \in R$).

Find an expression for $\Bbbk[x_1, \ldots, x_n]_{x_i}$ as a ring of polynomials. Is $\Bbbk[x_1, \ldots, x_n]_{x_i - x_j}$ isomorphic to $\Bbbk[x_1, \ldots, x_n]_{x_i}$? Here $\Bbbk$ is an arbitrary field and $1 \leq i < j \leq n$.

Problem 1. Recall that $\mathbb{P}^n_{\Bbbk} = \{[x_0 : \cdots : x_n] \mid x_i \in \Bbbk\} / \sim$ where $\sim$ is given by simultaneous scaling. We have a standard decomposition of the projective space into $(n+1)$ *affine charts*

$$\mathbb{P}^n_{\Bbbk} = \bigcup_{i=0}^{n} U_i,$$

where $U_i = U_{x_i} = \{[x_0 : \cdots : x_n] \in \mathbb{P}^n_{\Bbbk} \mid x_i \neq 0\}$.

On each U_i there are natural coordinates

$$(u_{0/i}, u_{1/i}, \ldots, u_{(i-1)/i}, u_{(i+1)/i}, \ldots u_{n/i}) = (\frac{x_0}{x_i}, \ldots, \frac{x_{i-1}}{x_i}, \frac{x_{i+1}}{x_i}, \ldots \frac{x_n}{x_i})$$

and we put $\mathcal{O}_i = \mathcal{O}_{U_i}$ to be the set of polynomial functions on U_i, i.e.

$$\mathcal{O}_{U_i} = \Bbbk[u_{0/i}, \ldots, u_{n/i}]$$

.

Denote by $U_{ij} = U_i \cap U_j$. From the perspective of U_i, this set is a complement of the zero-set $V(u_{j/i})$, and hence polynomial functions on it are given by $(\mathcal{O}_i)_{u_{j/i}}$. From the perspective of U_j we get $(\mathcal{O}_j)_{u_{i/j}}$.

(a) For $n = 2$ find a natural isomorphism between

$$\varphi_{ij} \colon (\mathcal{O}_i)_{u_{j/i}} \to (\mathcal{O}_j)_{u_{i/j}}.$$

(b) Generalize the formula to arbitrary n.

We will use the notation $\mathcal{O}_{ij}$ to denote the ring of functions on U_{ij}, which is given by any of the two isomorphic rings above.

Remark. Notice that each U_{x_i} can be treated as a set where x_i does not vanish. In a similar manner, one could define U_f for any homogeneous polynomial f, so our choice of the cover of $\mathbb{P}^n$ by affine charts is not canonical.

Problem 2. We say that a polynomial $f \in \Bbbk[x_0, \ldots, x_n]$ is *homogeneous of degree d* if it satisfies $f(\lambda x_0, \ldots, \lambda x_n) \equiv \lambda^d f(x_0, \ldots, x_n)$. A **projective variety** is a common zero-set $X = V(f_1, \ldots, f_r) \subset \mathbb{P}^n_{\Bbbk}$ of a finite collection of homogeneous polynomials.

(1) Show that any conic can be presented as a projective variety $C \subset \mathbb{P}^2_{\Bbbk}$.

(2) Consider the following affine variety $X \subset U_0$. By looking at the preimages of the corresponding polynomials, find a projective variety $\overline{X} \subset \mathbb{P}^n$ such that $\overline{X} \cap U_0 = X$. This variety is called *projective closure*.

(a) $X = V(u_{2/0}^2 - u_{1/0}^3) \subset U_0$, $n = 2$;

(b) $X = V(u_{2/0} - u_{1/0}^2, u_{2/0} + u_{1/0} - 2)$, $n = 2$;

(c) $X = V(u_{2/0}(u_{3/0} - 1), (u_{3/0} - u_{1/0}^3)(u_{3/0} - 1))$, $n = 3$.

(3) Assume $X \subset \mathbb{P}^n_{\Bbbk}$ is a projective variety. Explain how the same set can be treated as a projective variety inside $\mathbb{P}^N_{\Bbbk}$ for arbitrary $N > n$.

(4) Let $(\Bbbk[x_0, \ldots, x_n])_d$ be the set of degree d polynomials. Find a natural map

$$\mathrm{res}_i \colon (\Bbbk[x_0, \ldots, x_n])_d \to \mathcal{O}_{U_i}$$

such that for any polynomial P of degree d it holds that $V(P) \cap U_i = V(\mathrm{res}_i(P))$, i.e., the zero-set of the homogenous polynomial P within the affine chart U_i is given by the zero-set of $\mathrm{res}_i(P)$.

(5) Show that the set $X = \{[s^3 : st^2 : t^3] \in \mathbb{P}^2 \mid (s, t) \in \Bbbk^2 \setminus \{(0, 0)\}\}$ is a projective variety.

Problem 3 (Segre embedding). (0) If you would like to review some important concepts related to determinants of matrices, go to the last page at this point.

(1) Consider $\mathbb{P}^{nm+n+m}$ regarded as a projectivization of the set of $(n+1)\times(m+1)$ matrices. Show that the set of matrices of rank at most k constitutes a projective subvariety. *Hint: recall that the determinant function* det *can be used to detect the rank of a given matrix*

(2) Construct an algebraic map (with respect to homogeneous coordinates)

$$\varphi_{n,m}\colon \mathbb{P}^n \times \mathbb{P}^m \to \mathbb{P}^{nm+n+m},$$

which is injective and has rank 1 matrices as its image. This map is called *Segre embedding*.

(3) Show that $\mathrm{Im}(\varphi_{1,1}) \subset \mathbb{P}^3$ is a projective surface, and write down its equations explicitly.

Problem 4 (Veronese embedding). We define the embedding

$$\psi_d^n\colon \mathbb{P}^n \hookrightarrow \mathbb{P}^{N(d,n)}$$

by sending the point $[x_0\colon \ldots : x_N]$ to the point given by the tuple of all possible degree d monomials.

(1) Find $N(d,n)$ and show that ψ_d^n is indeed injective.

(2) When $n = 1$ the image is called *Veronese curve*. Show that $\mathrm{Im}(\psi_2^1)$ is a projective curve in $\mathbb{P}^2$ and find its degree. Here we have

$$\psi_2^1([x:y]) = [x^2 : xy : y^2].$$

(3) Find 3 different (with different zero-sets) quadratic homogeneous polynomials vanishing on the Veronese curve $\mathrm{Im}(\psi_3^1) \subset \mathbb{P}^3_{\Bbbk}$. Here we have

$$\psi_3^1([x:y]) = ([x^3 : x^2y : xy^2 : y^3]).$$

(4) The curve from the previous is example is called *twisted cubic*. What is so *cubic* about it? We say that the degree of a curve in $\mathbb{P}^n_{\Bbbk}$ is the number of times this curve intersects a generic hyperplane. Show that for $\Bbbk = \mathbb{C}$ the Veronese curve $\mathrm{Im}(\psi^1_3)$ intersects an arbitrary hyperplane 3 times (maybe with multiplicities).

(5) Consider the *Veronese surface* $\mathrm{Im}(\psi^2_2) \subset \mathbb{P}^5$ where

$$\psi^2_2([x : y : z]) = ([x^2 : xy : xz : y^2 : yz : z^2]).$$

Using coordinates $([w_0 : w_1 : w_2 : w_3 : w_4 : w_5])$ on $\mathbb{P}^5$ find 3 distinct quadratic equations vanishing on $\mathrm{Im}(\psi^2_2)$.

(6) Show that for arbirtary line $\ell \subset \mathbb{P}^2$ its image under ψ^2_2 is a conic curve in $\mathbb{P}^5$. To do so, show that this curve $\psi^2_2(\ell)$ can be given as a zero-set of three distinct linear polynomials and a quadratic polynomial (clearly, this implies that an arbitrary hyperplane intersects $\psi^2_2(\ell)$ in not more than 2 points, hence it is a degree 2 curve).

(7) Let us denote by $\pi(\ell)$ the projective plane in $\mathbb{P}^5$ containing the conic $\psi^2_2(\ell)$ (using the previous problem, it should not be hard to describe it in terms of the equation of ℓ). Show that the union of planes $\pi(\ell)$ is a cubic hypersurface in $\mathbb{P}^5$.

Hint: when does a homogeneous linear system of equations have a nontrivial solution?

2 Tensor product

Definition. Let V and W be two vector spaces over the field $\Bbbk$. Consider a (*massive*) vector space L with basis elements $v \otimes w$ for any $v \in V, w \in W$. Now consider a vector subspace R (of relations) spanned by

1. $(v_1 + v_2) \otimes w - v_1 \otimes w - v_2 \otimes w$;
2. $v \otimes (w_1 + w_2) - v \otimes w_1 - v \otimes w_2$;
3. $\lambda u \otimes w - \lambda(u \otimes w)$;
4. $u \otimes \lambda w - \lambda(u \otimes w)$.

We define the tensor product of V and W as a quotient $V \otimes W = L/R$.

Remark. In the same fashion, one defines the tensor product of two modules over a given ring.

Problem 1. 1. Show that the map $\hat{\otimes}\colon V \times W \to V \otimes W$ given by $(v, w) \mapsto v \otimes w$ is a well-defined *bilinear* map (i.e., it is linear in each of the two coordinates).

2. Show that the map $\hat{\otimes}$ satisfies the following *universal property*. For any *bilinear map* $F\colon V \times W \to Z$ for any vector space Z there exists a *unique* linear map $\hat{F}\colon V \otimes W \to Z$ such that
$$\hat{F} \circ \hat{\otimes} = F.$$

Problem 2. 1. Let $\dim_{\Bbbk}(V) = n$ and $\dim_{\Bbbk}(W) = m$. Express the dimension of the tensor product $V \otimes W$ in terms of the numbers n and m.

2. Any rank 1 free abelian group is isomorphic to $\mathbb{Z}$ (by definition), but there are 2 such isomorphisms! The choice of such an isomorphism is called *orientation* (note that it would be *really hard* for you to explain to an alien civilization what is right and what is left if you are only allowed to send text messages).

   Given two rank 1 free abelian groups O_1 and O_2, let us consider their tensor product $O_1 \otimes_{\mathbb{Z}} O_2$ (note that abelian groups are naturally modules over $\mathbb{Z}$). Show that $O_1 \otimes_{\mathbb{Z}} O_2$ is a rank 1 free abelian group. Show that orientations on O_1 and O_2 induce an orientation on $O_1 \otimes_{\mathbb{Z}} O_2$. Which pairs of orientations on O_1 and O_2 induce the same orientation on $O_1 \otimes_{\mathbb{Z}} O_2$?

3. Express the following tensor products of the abelian groups in more familiar terms: $\mathbb{Z} \otimes \mathbb{Z}/5$, $\mathbb{Z}/2 \otimes \mathbb{Z}/3$, $\mathbb{Z}/12 \otimes \mathbb{Z}/8$.

4. Solve the same problem for the following tensor products of modules over polynomial rings:

   (a) $\Bbbk[x, y] \otimes_{\Bbbk[x]} \Bbbk[x, z]$;

   (b) $\Bbbk[x, y]/((xy - y + x + 1)) \otimes_{\Bbbk[x,y]} \Bbbk[x, y]/((x - 1))$;

   (c) $\Bbbk[x, y, y^{-1}] \otimes_{\Bbbk[x,y]} \Bbbk[x, x^{-1}, y]$.

3 Review of determinants.

Definition. Given a matrix $A = \left(a_{ij}\right)_{1\leq i,j\leq n}$ we set its determinant to be the following function

$$\det(A) = \sum_{\sigma\in S_n} \operatorname{sgn}(\sigma) \prod_{i=1}^{n} a_{i\sigma(i)},$$

where $\operatorname{sgn}(\sigma)$ is the sign of the permutation $\sigma \in S_n$, i.e. the number of *inversions*.

Problem 1. (1) Show that the determinant of the diagonal matrix A is equal to the product of its entries.

(2) Show that $\det(AB) = \det(A)\det(B)$ for any two $n \times n$ matrices A and B.

(3) Using the above two properties, show that if $\det(A) = 0$ then A is not invertible, i.e. there is no B such that $AB = \mathrm{Id_n}$.

For later use, we recall that A is invertible if and only $\operatorname{Ker}(A) = \{0\}$.

(4) Consider polynomial $p_A(\lambda) = \det(A - \lambda\,\mathrm{Id}_n)$ for $\lambda \in \mathbb{C}$. Show that if $p_A(\lambda) = 0$ then there exists a vector $v_\lambda \in \mathbb{C}^n$ such that $A(v_\lambda) = \lambda v$. The λ itself is called an *eigenvalue*.

(6) Show that $\det(A)$ is the free coefficient of $p_A(\lambda)$.

(7) Using the fundamental theorem of algebra, conclude that if $\det(A) \neq 0$ then A is invertible.

Outcome: We just proved the following theorem: A is invertible if and only if $\mathrm{A} = 0$.

Problem 2. (0) The rank of an $n \times m$ matrix is defined to be the maximal number of linearly independent column vectors. Recall that there is an associated linear map $T(A)\colon \mathbb{R}^m \to \mathbb{R}^n$, and the above rank is clearly equal to the dimension of the image of $T(A)$, i.e.

$$\operatorname{rk}(A) = \dim(\operatorname{Im}(A)).$$

(1) Use determinant to say that $\operatorname{rk}(A) \leq n - 1$ for $A \in M_n(\mathbb{R})$.

(2) Use determinant to say that $\operatorname{rk}(A) \leq n - 2$ to for $A \in M_n(\mathbb{R})$.

(3) Use determinant to say that $\operatorname{rk}(A) \leq k$ to for $A \in M_{n\times m}(\mathbb{R})$.