Oleg Gleizer
prof1140g@math.ucla.edu

Brendan Connelly
brendanconnelly@ucla.edu

Geometry - Introduction to Linear Algebra

Preliminary - Warm Up

Problem 1. Prove that in a group of six people there are either three mutual friends or three mutual strangers.

Hint: Represent the people by the vertices of a regular hexagon. Connect two vertices by a red line segment if the corresponding pair of people are friends; otherwise, connect them with a blue line segment. Pick one vertex, say A. At least three line segments emanating from A have the same color. Consider the two cases and finish.

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1. What are vectors? How are they different than numbers?

What most people learn in high school is that a vector is a mathematical object with magnitude and direction. The most classic example of this is velocity, comprised of a speed and a direction of travel. But, what does this actually mean? What does it mean to have a direction? Or a magnitude?

Note 1. Vectors arise naturally in geometry and physics when describing quantities such as velocities and forces. Geometrically, a vector can be represented by a pointed arrow: the direction of the arrow indicates direction, and the length of the arrow represents magnitude (for example, speed). Two arrows are considered to represent the same vector if one can be obtained from the other by parallel translation. In other words, what matters is the "move," not where the arrow is placed.

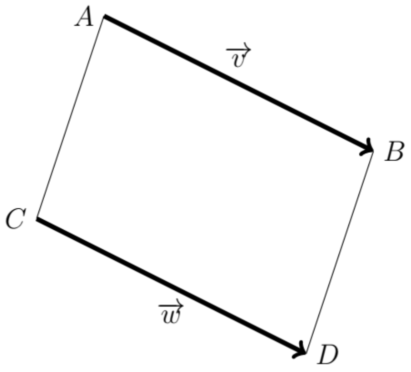


Figure 1: $\vec{v} = \vec{w}$

Example 1. A robot moves in the plane in three stages. How can we describe its *final position* using vectors?

Stage 1: it moves East at 3 m/s for 8 s. Stage 2: it moves North at 2 m/s for 5 s. Stage 3: it moves in the direction $(1, -2)$ (i.e. one unit East for every two units South) at speed 4 m/s for 3 s.

Each stage produces a displacement vector:

$$\vec{d_1} = 8 \cdot (3, 0) = (24, 0), \qquad \vec{d_2} = 5 \cdot (0, 2) = (0, 10).$$

For Stage 3 we first turn the direction $(1, -2)$ into a *unit direction*:

$$\|(1, -2)\| = \sqrt{1^2 + (-2)^2} = \sqrt{5}, \qquad \text{unit direction} = \left(\frac{1}{\sqrt{5}}, \frac{-2}{\sqrt{5}}\right).$$

Speed × time is $4 \cdot 3 = 12$, so

$$\vec{d_3} = 12\left(\frac{1}{\sqrt{5}}, \frac{-2}{\sqrt{5}}\right) = \left(\frac{12}{\sqrt{5}}, \frac{-24}{\sqrt{5}}\right).$$

Therefore the total displacement is the vector sum

$$\vec{d} = \vec{d_1} + \vec{d_2} + \vec{d_3} = \left(24 + \frac{12}{\sqrt{5}},\ 10 - \frac{24}{\sqrt{5}}\right).$$

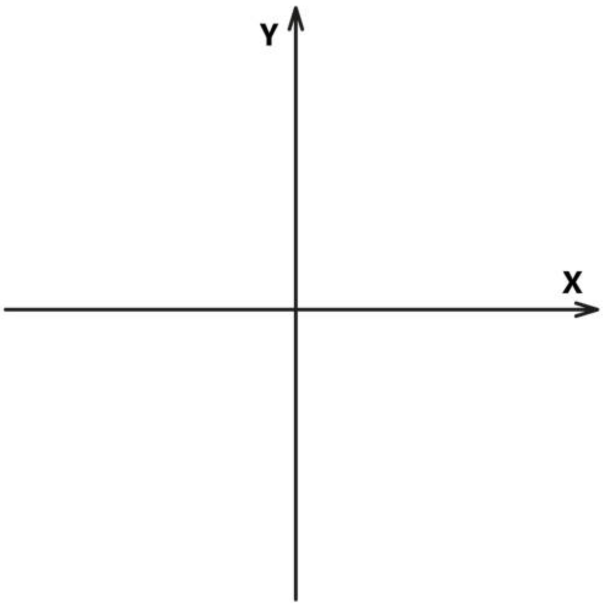
This illustrates the point: the final outcome is naturally an ordered pair, and combining stages is exactly *vector addition*. Scaling motion (changing time or speed) is *scalar multiplication*.

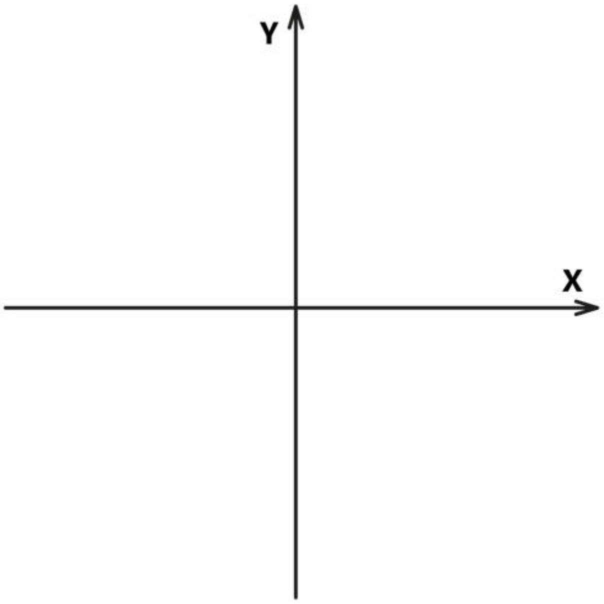
Note 2. Important: many different arrows can represent the same "move." For example, the arrow from $(0, 0)$ to $(3, 4)$ and the arrow from $(10, -2)$ to $(13, 2)$ both represent the same move $(3, 4)$. So in math, we usually treat the vector as the move itself, not the specific arrow sitting in a specific place.

Problem 2. Let

$$\vec{v} = (2, 1), \quad \vec{w} = (-1, 2), \quad \vec{u} = (1, -3).$$

1. On the same set of axes, draw the vectors $\vec{v}$, $\vec{w}$, $\vec{u}$, and $2\vec{v}$ with their tails at the origin.
2. Compute $\vec{v} + \vec{w}$ and draw this vector.
3. Do the same for $\vec{w} + \vec{v}$. What do you notice?
4. Compute $(\vec{v} + \vec{w}) + \vec{u}$ and draw the resulting vector.
5. Do the same for $\vec{v} + (\vec{w} + \vec{u})$. What do you notice?





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Note 3. We can observe a few properties we want our mathematical definition of vectors to satisfy from the above examples

- add vectors
- addition is associative, meaning ____________________
- addition is commutative, meaning ____________________

Problem 3. Can you list at least one more property that vectors should have based on the examples we have done so far? Can you say something about zero? Or about what must be true about numbers?

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We need to define what we mean by a "number"? In the above cases, numbers are simply real numbers, we can multiply any of our vectors by any real number, and that scaled vector is a mathematical quantity that makes sense. Generally, all vectors are defined over an algebraic structure called a field. The real numbers are a field. We will simply define all our vector spaces over the field, $\mathbb{R}$. It helps to remember that real numbers satisfy the following properties:

Note 4 (Field axioms for $\mathbb{R}$)**.** A field is a set (here, $\mathbb{R}$) with two operations, addition and multiplication, such that:

1. (Addition is commutative) For all $a, b \in \mathbb{R}, \quad a + b = b + a$.
2. (Addition is associative) For all $a, b, c \in \mathbb{R}, \quad (a + b) + c = a + (b + c)$.
3. (Additive identity) There exists a number $0 \in \mathbb{R}$ such that for all $a \in \mathbb{R}, \quad a + 0 = a$.
4. (Additive inverses) For every $a \in \mathbb{R}$ there exists $-a \in \mathbb{R}$ such that $\quad a + (-a) = 0$.
5. (Multiplication is commutative) For all $a, b \in \mathbb{R}, \quad ab = ba$.
6. (Multiplication is associative) For all $a, b, c \in \mathbb{R}, \quad (ab)c = a(bc)$.
7. (Multiplicative identity) There exists a number $1 \in \mathbb{R}$ with $1 \neq 0$ such that for all $a \in \mathbb{R}, \quad a \cdot 1 = a$.
8. (Multiplicative inverses) For every $a \in \mathbb{R}$ with $a \neq 0$ there exists $a^{-1} \in \mathbb{R}$ such that $a \cdot a^{-1} = 1$.
9. (Distributive law) For all $a, b, c \in \mathbb{R}, \quad a(b + c) = ab + ac$.

These are exactly the properties of real numbers you use when you do algebra. **There are a lot of axioms! Take these to mean when you are working only with real numbers, you can do all the things you are used to!**

Note on $\mathbb{R}$. Formally, as you are likely familiar, $\mathbb{R}$ is not just a field: it is an *ordered field* (it has a total order compatible with $+$ and $\cdot$), it is *Archimedean*, and it is *complete* (every nonempty set bounded above has a least upper bound). Equivalently, $\mathbb{R}$ can be characterized as the unique (up to isomorphism) *complete ordered field*, i.e. the (essentially unique) ordered completion of $\mathbb{Q}$. We will not prove these facts here. **We only need the field axioms for $\mathbb{R}$ in this packet.**

Problem 4. Prove that $0 \cdot a = 0$ for all $a \in \mathbb{R}$. *Hint: Consider* $0 + 0 = 0$

Problem 5. Let $a \in \mathbb{R}$. Prove that

$$(-1) \cdot a = -a,$$

where $-a$ denotes the additive inverse of a.

Problem 6. Let $a, b \in \mathbb{R}$.

- Prove that
$$(-a) \cdot b = -(ab).$$
- Then conclude
$$(-a) \cdot (-b) = ab.$$

2. What is a Vector Space (and what actually are vectors?)

Definition 1. *A* vector space over $\mathbb{R}$ *is a set* V *together with two operations*

$$+ : V \times V \to V \qquad \textit{and} \qquad \cdot : \mathbb{R} \times V \to V,$$

called vector addition *and* number multiplication*, such that for all* $u, v, w \in V$ *and all* $a, b \in \mathbb{R}$*, the following axioms hold.*

Definition 2 (Vector Space Axioms)**.** *Let V be a vector space over $\mathbb{R}$. For all $u, v, w \in V$ and all $a, b \in \mathbb{R}$:*

1. *(Commutativity of addition) $u + v = v + u$.*
2. *(Associativity of addition) $(u + v) + w = u + (v + w)$.*
3. *(Additive identity) There exists an element $\vec{0} \in V$ such that $v + \vec{0} = v$ for all $v \in V$.*
4. *(Additive inverses) For every $v \in V$ there exists an element $-v \in V$ such that $v + (-v) = \vec{0}$.*
5. *(Distributivity over vector addition) $a(u + v) = au + av$.*
6. *(Distributivity over number addition) $(a + b)v = av + bv$.*
7. *(Compatibility of number multiplication) $a(bv) = (ab)v$.*
8. *(number identity) $1v = v$.*

Note 5. The symbol $\vec{0}$ is the *zero vector* in V. This is opposed to the zero number, in our case, $0 \in \mathbb{R}$.

We will consider a few examples of vector spaces. To prove that a set with specific operations describes a vector space, one needs to show all 8 of the above axioms. In practice, this is tedious and takes a while. **Choose one axiom of the eight vector space axioms to prove for each of the following examples:**

Example 2 ($\mathbb{R}^n$)**.** Let $V = \mathbb{R}^n$, the set of all ordered lists of n real numbers:

$$(x_1, x_2, \ldots, x_n).$$

Remember, to define a vector space, we need to define addition and multiplication.

We add vectors by adding their coordinates, and we multiply a vector by a real number by multiplying each coordinate:

$$(x_1, \ldots, x_n) + (y_1, \ldots, y_n) = (x_1 + y_1, \ldots, x_n + y_n),$$

$$a(x_1, \ldots, x_n) = (ax_1, \ldots, ax_n).$$

These rules behave exactly the way vector addition and number multiplication should. The zero vector is $(0, \ldots, 0)$, and every vector has an opposite (its negative). Because all the vector space axioms follow from properties of real numbers applied coordinate-by-coordinate, $\mathbb{R}^n$ is a vector space.

Example 3 (Polynomials of degree 2 or less $P_2(\mathbb{R})$ - quadratic, linear, and constant functions)**.** Let $V = P_2(\mathbb{R})$ be the set of all real polynomials of degree at most 2, meaning

$$P_2(\mathbb{R}) = \{\, a_0 + a_1x + a_2x^2 \mid a_0, a_1, a_2 \in \mathbb{R} \,\}.$$

So each "vector" in this vector space is a polynomial like

$$p(x) = a_0 + a_1x + a_2x^2.$$

We add polynomials by adding their formulas, and we multiply a polynomial by a real number by multiplying every coefficient:

$$(p + q)(x) = p(x) + q(x), \qquad (ap)(x) = a\ (p(x))\,.$$

The zero vector is the zero polynomial $0(x) = 0$, and the negative of a polynomial is the polynomial obtained by negating each coefficient.

Problem 7. What do you notice about how $\mathbb{R}^3$ compares to $P_2(\mathbb{R})$. Think about the vectors x and $\langle 0, 1, 0\rangle$? Can you find some way to relate vectors in one vector space to another?

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3. Some Properties of Vector Spaces

Theorem 1 (The zero vector is unique! There is only one zero vector!)**.** If $\vec{z} \in V$ satisfies $\vec{v} + \vec{z} = \vec{v}$ for all $\vec{v} \in V$, then $\vec{z} = \vec{0}$.

Problem 8. Prove the theorem above. *Hint: take* $\vec{v} = 0$

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One important fact we can prove easily that helps with later proofs is the following:

Theorem 2 (Cancellation law)**.** If $\vec{a}, \vec{b}, \vec{c} \in V$ and $\vec{a} + \vec{c} = \vec{b} + \vec{c}$, then $\vec{a} = \vec{b}$.

Problem 9. Prove the theorem above.

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This the helps us prove an even weaker version of the uniqueness of the zero vector

Problem 10 (Uniqueness of the zero vector)**.** If $\vec{z} \in V$ satisfies $\vec{v} + \vec{z} = \vec{v}$ for some $\vec{v} \in V$, then $\vec{z} = \vec{0}$. Prove this.

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Problem 11. Prove that if $a\vec{v} = \vec{0}$ for some number $a \in \mathbb{R}$ and vector $\vec{v} \in V$, then either $a = 0$ or $\vec{v} = \vec{0}$.

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Problem 12. Prove that $0v = 0$ for $v \in V$. *Consider your proof in question 3.*

4. Vector Subspaces

As mentioned, it is tedious to prove a set with given operations is a vector space. However, if we have some subset of a vector space, the subset may be a vector space by itself. This is true as long as the following hold

Definition 3. *Let V be a vector space over $\mathbb{R}$. A* ***nonempty*** *subset $W \subseteq V$ is a* vector subspace *(or* subspace*) of V if for all $v, w \in W$ and all $\lambda \in \mathbb{R}$,*

1. $v + w \in W$ *(closure under addition)*

2. $\lambda v \in W$ *(closure under number multiplication)*

It is useful to have this definition, as it gives us many more vector spaces (again, each subspace is a vector space in itself). After checking the subspace axioms, we inherit all the vector space properties from the larger space.

Problem 13. Sometimes the above definition requires that $\vec{0} \in W$ instead of just that the subset is non-empty. Show that under this definition, if W is a subspace, it must contain $\vec{0}$. This proves the equivalence of both ways of defining a vector subspace.

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Problem 14. Think about the line $y = 3x$. In set notation, consider the set

$$W = \{(x, y) \in \mathbb{R}^2 \mid y = 3x\}.$$

1. Show that W is nonempty.
2. Show that if $(x_1, y_1), (x_2, y_2) \in W$, then their sum is also in W.
3. Show that if $(x, y) \in W$ and $\lambda \in \mathbb{R}$, then $\lambda(x, y) \in W$.

Conclude that W is a vector subspace of $\mathbb{R}^2$.

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Problem 15. Consider the set

$$U = \{(x, y) \in \mathbb{R}^2 \mid y = 3x + 1\}.$$

1. Show that U is nonempty.
2. Check whether U is closed under addition.
3. Check whether U is closed under number multiplication.

Is U a vector subspace of $\mathbb{R}^2$? Explain which property fails.

Problem 16. Let P_2 be the vector space of real polynomials of degree at most 2. Consider the subset

$$W = \{(x + 1)q(x) \mid q(x) \in P_1\}.$$

Prove that W is a subspace of P_2. *Note that P_1 is also a vector space (and hence is closed)*

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Theorem 3. The set $C(\mathbb{R})$ of all continuous functions $f : \mathbb{R} \to \mathbb{R}$, equipped with pointwise addition and scalar multiplication,

$$(f+g)(x) \;=\; f(x)+g(x), \qquad (\lambda f)(x) \;=\; \lambda f(x),$$

is a vector space over $\mathbb{R}$. *Check this as an additional exercise if you are interested/have time. It does require basic facts from calculus/analysis, e.g., sum of continuous functions is continuous and so on*

Problem 17. Let

$$W = \{\, f \in C(\mathbb{R}) \mid f(0) = 0 \,\}.$$

1. Show that W is nonempty.
2. Show that if $f, g \in W$, then $f + g \in W$.
3. Show that if $f \in W$ and $\lambda \in \mathbb{R}$, then $\lambda f \in W$.

Conclude that W is a vector subspace of $C(\mathbb{R})$.

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Problem 18 (A non-example)**.** Let

$$U = \{\, f \in C(\mathbb{R}) \mid f(0) = 1 \,\}.$$

1. Show that U is nonempty.
2. Check whether U is closed under addition.
3. Check whether U is closed under scalar multiplication.

Is U a vector subspace of $C(\mathbb{R})$? Explain precisely which property fails.

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5. Bonus

We will discuss more geometric applications of vectors in later weeks, but here is a bonus problem:

Problem 19. Use vectors to show that the diagonals of a parallelogram split each other in halves.

1. Give a vector proof in $\mathbb{R}^2$.
2. Generalize the argument and prove the same statement in $\mathbb{R}^3$ and $\mathbb{R}^4$.