Oleg Gleizer
prof1140g@math.ucla.edu

Seth Tang
sethtang001@g.ucla.edu

Set theory and its fun applications

Near the end of the last packet, we introduced the concept of an infinite set. For finite sets, comparing cardinality is straightforward. For infinite sets, we need a more careful approach.

If two sets A and $\mathcal{F}(A)$ (finite or infinite) have the same cardinality, we say A and $\mathcal{F}(A)$ are **equipotent**. We write this as $A \sim B$. Recall Problem 31 from the last packet: for finite sets, if we can establish a bijection between them, then they have the same cardinality. Interestingly, this connection between bijections and cardinality also carries over to infinite sets.

Problem 1 *Consider the following:*

1. *Is* $f(x) = 2x$ *bijective?*
2. *Is it possible that* $\mathbb{N} \sim 2\mathbb{N}$ *is true?*
3. *Is it appropriate to say that if we can establish a bijection between two sets, then they are equipotent?*

This fascinating connection also gave birth to one of the most interesting math mind teasers: the **infinite hotel problem**! Suppose you are visiting a hotel. Each door is labeled by a natural number, with no repetition. One guest checks into exactly one room. You have been hired as a business consultant to solve room-assignment issues.

Problem 2 *Suppose two buses arrive, each carrying infinitely many passengers. How would you assign rooms efficiently? What if four buses arrive? (Tip: Which earlier problem is this similar to?)*

Consider the following visualization of the four buses. Within each parenthesis, the passenger's bus number and seat number within the bus are provided.

(1,1), (1,2), (1,3) (1,4).................
(2,1), (2,2), (2,3) (2,4).................
(3,1), (3,2), (3,3) (3,4).................
(4,1), (4,2), (4,3) (4,4).................

Problem 3 *Suppose infinitely many buses arrive, each with infinitely many passengers. How would you assign rooms efficiently? (Tip: Try to visualize it in a grid.)*

Problem 4 *The manager suggests the following method: let the bus number* n *correspond to the* n*th prime number, and for each* k*th passenger on the bus, we assign room* p_n^k*. From a business perspective, is this a good idea?*

Problem 5 *A mathematician suggested the following: let* (n, m) *denote the* m*th passenger on the* n*th bus. The assigned room number* $R(n, m)$ *is defined* *as:*

$$R(n,m) = \begin{cases} (n-1)^2 + m, & \textit{if } n \geq m, \\ m^2 - n + 1, & \textit{if } n < m. \end{cases}$$

Using the visual representation, is there any pattern to the room assignment? How does it compare to the manager's plan? (Ask the instructor if you are stuck.)

Problem 6 *Suppose all rooms with square-number labels are under renovation (i.e., $1, 4, 9, 16, \ldots$). Can we still tackle the aforementioned problem?*

As these problems have shown, we observe that $\mathbb{N} \sim 4\mathbb{N}$ and $\mathbb{N} \sim \mathbb{N} \times \mathbb{N}$ are both valid, despite human intuition suggesting otherwise.

If there exists a bijection $f : \mathbb{N} \to A$ (that is, $\mathbb{N} \sim A$), then A is said to be **countably infinite**. The converse is true as well.

Problem 7 *Based on this definition, what are some examples of countable sets? Are integers countable? Are rational numbers countable? How about real numbers? (Tip: For rational numbers, try visualizing them in a grid.)*

We shall now introduce a related theorem:

Theorem 1 *If every element of a set A can be identified uniquely using a finite collection of natural numbers, then A is either finite or countable.*

Problem 8 *Complete the proof of Theorem 1. Consider the manager's idea of assigning rooms using prime numbers. Here are some hints:*

1. *Describe an element X in A as $(x_1, x_2, x_3, x_4, \ldots, x_n)$.*
2. *List the prime numbers as $(p_1, p_2, p_3, p_4, \ldots, p_n, \ldots)$.*

(Tip: Can we use the position and the prime in that position in a meaningful way?)

Problem 9 *Can π be identified uniquely by a finite collection of natural numbers? (Tip: This is a little tricky. Ask your instructor if you are stuck.)*

Problem 10 *Assume the real numbers are infinite and not countable. What does this imply?*

If you concluded that the real numbers are countable, then the following proof may change your mind. Suppose $\mathbb{N}$ and $\mathbb{R}$ are bijective. That is, for every element in $\mathbb{R}$, we can match it to exactly one element in $\mathbb{N}$. Now suppose we line up all elements of $\mathbb{R}$ in rows as follows:

$$1 \Rightarrow 0.100000...$$
$$2 \Rightarrow 1.305069...$$
$$3 \Rightarrow 9.842093...$$

Now let us construct a new element of $\mathbb{R}$ that is not mapped to any natural number. For the nth position, we check the nth digit of the nth row, denoted by k. Set the nth digit to be any digit not equal to k, and proceed in this way for all positions. Here is one example:

$$\text{New number} \Rightarrow 1.26...$$

At the end, we have produced a real number that is different from every listed real number. It is also not mapped to any number in $\mathbb{N}$, since we assumed the list already exhausted all of $\mathbb{N}$. An equally useful version of this argument uses only 0s and 1s (you may wish to remember this fact!)

Clearly, we have shown that there exists at least one set such that no bijection exists between it and the natural numbers. Therefore, **uncountably infinite** sets exist.

Now let us return to countable and uncountable sets. Compared to a finite set, both countable and uncountable sets are larger. But between the two infinite types, which one has greater cardinality?

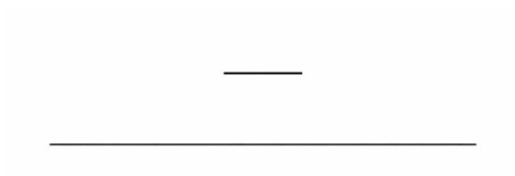
Problem 11 *Here are some useful things to consider. Consider any uncountable set A.*

1. *Is it possible to construct a countable proper subset S of A?*
2. *Recall that for any proper subset $S \subset A$, we have $|S| \leq |A|$.*

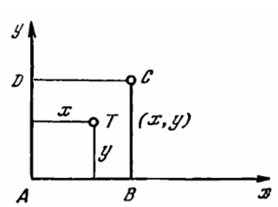
In our case, can $|S| = |A|$ be true? (Tip: For part 1, the answer may be simpler than you imagine.)

We now consider some geometric examples. If we imagine the natural numbers as numbered rooms in a hotel, then we can visualize the real numbers as points on a line (the number line).

Problem 12 *Consider this picture. How would you establish a bijection between a longer line segment and a shorter line segment?*



Problem 13 *Consider this picture. How would you establish a bijection between a square and a line? (Tip: The answer is less pictorial.)*



Problem 14 *How would you establish a bijection between a curve of infinite length and a line? (Tip: The answer is very pictorial. Ask the instructor if you are stuck.)*

In summary, countable infinity corresponds to sets that can be put in bijection with the natural numbers, while uncountable infinity corresponds to sets that can be put in bijection with a line (equivalently, the real numbers).

Problem 15 *Is there a set of largest cardinality? (No proof is needed yet.)*

The proof of the previous problem is actually quite interesting, and it is worth investigating. Let $\mathcal{F}(A)$ be defined as such.

$$\mathcal{F}(A) \coloneqq \{\, f \mid f : A \to \{0,1\} \,\}.$$

The simple indicator function is a good example: for each $k \in A$,

$$f_k(x) = \begin{cases} 0, & \text{if } x \neq k, \\ 1, & \text{if } x = k. \end{cases}$$

However, the rules can be arbitrary. We do not need to list every possible function from A to $\{0,1\}$. For example, the function $g(x) = 0$ for all x also belongs to $\mathcal{F}(A)$.

The question now is: can A and $\mathcal{F}(A)$ be the same size? Suppose they are. Then every function in $\mathcal{F}(A)$ corresponds to exactly one element $x \in A$. For each $x \in A$, choose one function in $\mathcal{F}(A)$ and label it f_x.

Problem 16 *Now let us define*

$$h(x) = 1 - f_x(x)$$

(NOTE: We are not necessarily using the simple indicator function.) In other words, for an input $x \in A$, if $f_x(x) = 0$ then $h(x) = 1$, and if $f_x(x) = 1$ then $h(x) = 0$.

1. *Does $h(x)$ belong to $\mathcal{F}(A)$? If so, what does this imply? If h maps to exactly one element of A, what does that element map to?*

2. *Observe the following: $h(x) = f_k(x)$. Expand the equation. What happens if we set $x = k$? What contradiction does this cause? (Ask the instructor if you are stuck.)*

Thus, it is impossible for a bijection to exist between sets A and $\mathcal{F}(A)$. Therefore, they cannot be the same size. We also know $|B| \geq |A|$. This is because we can map each element of A to its simple indicator function, and those indicator functions form a subset of $\mathcal{F}(A)$. Therefore, A is strictly smaller than $\mathcal{F}(A)$. And since A is generalized as any set, no set of largest cardinality exists.

If we move beyond simple indicator functions, there are many applications of **generalized indicator functions**.

Problem 17 *Let $A = \{2, 3, 9, 6\}$. Write an indicator function representing the subset $C = \{3, 9, 6\}$.*

Recall Problem 11 from the last packet. An indicator function can represent any subset of a larger set. In particular, for any set A, the set of all subsets of A, called the power set, is equipotent to the set $\mathcal{F}(A)$.

As a final topic on infinity, let us explore some arithmetic identities. It will be fun to interpret them using our earlier examples of hotel rooms and line bijections. Let $\aleph_0$ denote the cardinality of countable sets, let c denote the cardinality of uncountable sets, and let f denote cardinalities larger than c.

$$n + \aleph_0 = \aleph_0 \tag{3.38}$$

$$\aleph_0 + \aleph_0 = \aleph_0 \tag{3.39}$$

$$\aleph_0 + \mathfrak{c} = \mathfrak{c} \tag{3.40}$$

$$\mathfrak{c} + \mathfrak{c} = \mathfrak{c} \tag{3.41}$$

$$\mathfrak{c} + \mathfrak{f} = \mathfrak{f} \tag{3.42}$$

Problem 18 *For 3.38 and 3.39, how can we utilize the infinite hotel problem?*

Problem 19 *For 3.40 and 3.41, how can we utilize the line-bijection problem? (Tip: 3.41 may be easier than 3.40.)*

Problem 20 *For 3.42, what reasoning can we offer? (Talk with the instructors about this: they are curious what you think!)*

The idea of an indicator function is a powerful tool. It has many applications. It can even solve problems that normally require higher-level techniques. This is possible because of the power of 0s and 1s. We will include an example at the end of the packet for fun.

Now we explore a new topic. As we have learned and applied earlier, functions, correspondence, mapping, and transformation are structurally similar ideas. In each case, we are establishing relationships between sets.

Much has been discussed about injectivity, surjectivity, and bijectivity, so we now move on to some shorter examples. (In the next two examples, the domain and range are $\mathbb{R}$.)

Problem 21 *Consider the famous Dirichlet function:*

$$f(x) = \begin{cases} 1, & x \in \mathbb{Q}, \\ 0, & x \in \mathbb{R} \setminus \mathbb{Q}. \end{cases}$$

Describe it in terms of the aforementioned concepts.

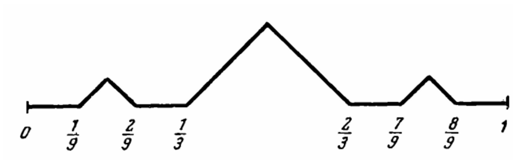
Problem 22 *Consider this function:*

$$f(x) = 0 \quad \textit{for all } x \in \mathbb{R}.$$

Describe it in terms of the aforementioned concepts.

Problem 23 *Suppose we restrict the range of the Dirichlet function to the set* $\{0, 1\}$*, and restrict its domain to the set* $\{1, \pi\}$*. Describe it now.*

Consider the interval $[0, 1]$ representing a campground. Tent 1 occupies the open interval $\left(\frac{1}{3}, \frac{2}{3}\right)$. Tent 2 occupies $\left(\frac{1}{9}, \frac{2}{9}\right)$, while Tent 3 occupies $\left(\frac{7}{9}, \frac{8}{9}\right)$. In the same pattern, smaller and smaller tents are erected on the remaining intervals. We call all uncovered points **wet points**.



Problem 24 *Consider all wet points at the edges of the tents (so far* $1/3, 2/3, 1/9, 7/9,$ *etc.). Is this set countable or uncountable? (Tip: Think about rational numbers.)*

Interestingly, the set of all wet points is actually uncountable. To show this, we will use additional tools. Consider ternary representation:

> It is easy to learn how to change the representation of a number whose ternary representation is
>
> $$0.02020202\ldots$$
>
> It is represented in the decimal system by the infinite geometric progression:
>
> $$\frac{2}{3^2} + \frac{2}{3^4} + \frac{2}{3^6} + \ldots \tag{4.4}$$
>
> The sum of this progression is $\frac{1}{4}$. Thus,
>
> $$\tfrac{1}{4} = 0.020202\ldots \tag{4.5}$$

Problem 25 *Consider all wet points at the edges of the tents. What are their ternary representations? What do you notice? (Tip: Observe that* $1/3$ *can be expanded as* $0.1000\ldots$ *in base 3.)*

Pay special attention to the presence of 1s in ternary representations. Notice that all points written as $0.1XYZ\ldots$ are covered by the first tent. All points written as $0.01XYZ\ldots$ are covered by the second tent. All points written as $0.21XYZ\ldots$ are covered by the third tent.

This pattern is useful. If we extrapolate it, we eventually discover that any ternary representation containing a 1 in any position will eventually become either (1) a countable wet point, or (2) a point covered by a tent.

Problem 26 *Consider the reasoning behind the previous statement. Why do all points with ternary representations containing a 1 have this property? (Tip: Ask the instructor if you are stuck.)*

Problem 27 *How would you describe all ternary representations that do not contain 1? Is it possible to map this set of real numbers to another set that is also uncountable? (Tip: Consider binary numbers.)*

Problem 28 *Consider the same tent setting scenario, except we set up tents taking up* $1/2$ *of the interval* $[0, 1]$*, starting with* $(0.5, 1)$*. Is it possible to create an uncountable set of wet points?*

Problem 29 *What total space do all the tents cover? (Ask the instructor if you are stuck.)*

Here are some additional problems: Consider the following

$$\aleph_0 \cdot \aleph_0 = \aleph_0 \tag{3.47}$$

$$\aleph_0 \cdot c = c \tag{3.48}$$

$$c \cdot c = c \tag{3.49}$$

$$c = 2^{\aleph_0} \tag{3.50}$$

Problem 30 *Using similar methods as earlier, reason why these identities are valid.*

Problem 31 *Show that among any six different two-element subsets of a set of cardinality five, there are two which are disjoint.*