Introduction to Set Theory

UCLA Math Circle Advanced 1B

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A **set** is a clearly defined collection of distinct objects.

Note that this is not really a definition. To define means to explain in simpler terms. Instead, all we do is replacing one word, a *set*, by another, a *collection*. Plus, the meaning of the words "to clearly define" is not clearly defined. The problem is that the notion of a set is as fundamental as it is deep. It is impossible to explain it in simpler terms. The best we can do at the moment is to show a bunch of examples.

The set F of all the factors of the number 6 consists of four elements: 6, 3, 2, and 1.

$$F = \{6, 3, 2, 1\}$$

Note that we use curvy braces, $\{*\}$, to write down the set **elements**. In a set, neither the order of elements nor duplicate elements matter, so we could equivalently say:

$$F = \{1, 6, 2, 3\} \qquad F = \{6, 3, 3, 2, 1\}$$

Problem 1. *What is the set of colors of the United States flag?*

Now let us list the elements of the set F in the increasing order:

$$F = (1, 2, 3, 6)$$

As you see, we have switched from braces to parentheses, $(*)$. Parentheses are used to write down elements of an *ordered set*, or a **list**. In a list, the order of the elements does matter. For example, let us consider two sets and two lists of letters:

$$S_1 = \{C, A, T\} \qquad L_1 = (C, A, T)$$
$$S_2 = \{A, C, T\} \qquad L_2 = (A, C, T)$$

Then $S_1 = S_2$, but $L_1 \neq L_2$. The fact that the letter A is an element of the set S_1 is denoted as

$$A \in S_1$$

As B is not an element of the set S_1, we write

$$B \notin S_1$$

Now let $S_3 = \{A, C\}$. One can see that every element of S_3 is also an element of S_1. In this case, S_3 is called a **subset** of S_1. In mathematical language, we write this as

$$S_3 \subseteq S_1$$

Now let's look at some examples of sets:

The set $\mathbb{N} = \{1, 2, 3, ...\}$ is called the set of **natural numbers**.
Other math textbooks/worksheets may define $\mathbb{N} = \{0, 1, 2, 3, ...\}$, with $0 \in \mathbb{N}$, but this is a matter of notational convention.

The set $\mathbb{Z} = \{..., -3, -2, -1, 0, 1, 2, 3, ...\}$ is called the set of **integers**.
We clearly see that $\mathbb{N}$ is a subset of $\mathbb{Z}$. In math notation, we write this as $\mathbb{N} \subseteq \mathbb{Z}$.

A set without any elements is called the **empty set** and is denoted as $\varnothing$ or $\{\}$.
Note that we call it the empty set and not a empty set since the empty set is unique.

A subset of a set is called a **proper subset** if it is not equal to the original set. For example, the set S_3 we have considered above is a proper subset of the set S_1. In mathematical language, we write this as $S_3 \subset S_1$.

Problem 2. *Is* $\{1, 1\}$ *a valid set?*

Problem 3. *Let* $S_1 = \{C, A, T\}$ *and* $S_2 = \{A, C, T\}$. *Is* S_1 *a subset of* S_2*? Is* S_2 *a proper subset of* S_2*?*

Problem 4. *Is there a set that is a subset of any set? Is there a set that is a proper subset of any set?*

Problem 5. *Write down all proper subsets of the set of colors of the U.S. flag.*

Let $m \in \mathbb{Z}$ and $n \in \mathbb{N}$. The set $\mathbb{Q}$ of all the fractions $\frac{m}{n}$ in the reduced form (a.k.a. in lowest terms) is called the set of **rational numbers**. Recall that the fraction $\frac{m}{n}$ is in the reduced form if $|m|$ and n have no common factors except 1. For example, $\frac{2}{3} \in \mathbb{Q}$.

Problem 6. *Prove that* $\mathbb{Z} \subset \mathbb{Q}$.

The number of elements in a set is called its **cardinality**. In math language, the cardinality of a set A is denoted as either $|A|$ or *card*(A). We will use the first notation in the handout. For example, $|S_1| = 3$ for the set $S_1 = \{C, A, T\}$ considered earlier.

Problem 7. *Let U be the set of states in the United States. What is $|U|$?*

Problem 8. *What is $|\varnothing|$? What is $|\{\varnothing\}|$?*

Problem 9. *What is* $\Big|\big\{\varnothing, a, b, \{a\}, \{a, b\}, \{a, a\}, \{b, a, b\}\big\}\Big|$*?*

Problem 10. *Let $A = \{a, b, c\}$ be a set and $A' = (a, b, c)$ be a list. For any set S, the **indicator function** $\mathbf{1}_S(x)$ returns 1 if $x \in X$ and 0 if $x \notin X$. For example, if $B = \{a, c\}$, then we have $\mathbf{1}_B(a) = 1,\ \mathbf{1}_B(b) = 0,\ \mathbf{1}_B(c) = 1$. The function always outputs a **bit** (0 or 1), so given a list A' of elements in A, subset B corresponds to an ordered **bit string** $\mathbf{1}_B(a), \mathbf{1}_B(b), \mathbf{1}_B(c) = 101$. Write two other subsets of A and their corresponding bit strings.*

Problem 11. *How many subsets does a set A have if $|A| = n$ for any nonnegative integer n? Prove this using the above terminology. (Hint: start by setting n to a small value, like 3).*

The set of the elements hat belong to the sets A and B is called the **intersection** of A and B and is denoted as $A \cap B$. Let us rewrite this definition completely in the math language:

$$A \cap B := \{x : x \in A \text{ and } x \in B\}$$

In this mathematical sentence, := means " is defined as" and : means "such that". Translating back to English, *the intersection of the sets A and B is defined as the set of the elements x such that x belongs to A and x belongs to B.*

Problem 12. *Let A be the set of all even numbers, and let B be the set of all integers divisible by 3. What is $A \cap B$?*

Problem 13. *What is $A \cap \varnothing$ for any set A?*

Notice how we used a specific mathematical notation to logically define an intersection of two sets, specifically the form:

$$\text{set} := \{\text{type of element in the set} : \text{condition that element must satisfy}\}$$

For more complex sets, our conditions may include logical operators (like "and" and "or") but also phrases used in formal logic like $\exists$ (there exists) and $\forall$ (for all). For instance, we can formally define the set of all even numbers A with the following:

$$A := \{2k : k \in \mathbb{Z}\}$$

but we also could have said

$$A := \{k : 2 \mid k\}$$

where $2 \mid k$ means 2 *divides* k.

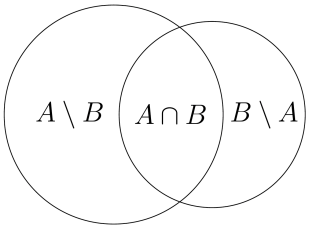
Problem 14. *Given a set A, the **power set** of A, $\mathcal{P}(A)$, is the set of all possible subsets of A. Formally define this.*

The following is a definition of **union** of two sets, written down in the math language.

$$A \cup B := \{x : x \in A \text{ or } x \in B\}$$

Problem 15. *Translate the above math into English.*

The **difference** of the sets A and B, the set $A \setminus B$, is the of all the elements of the set A that do not belong to the set B. The following picture, called a **Venn diagram**, helps to visualize the definition:



We can also make Venn diagrams with 3 or more sets.

Problem 16. *Let A be the set of spectators at a basketball game. Let B be the set of all people at the game (spectators, coaches, players, etc.) wearing caps. Describe in your own words the set $A \setminus B$.*

Problem 17. *Write the definition of $A \setminus B$ in our formal math notation.*

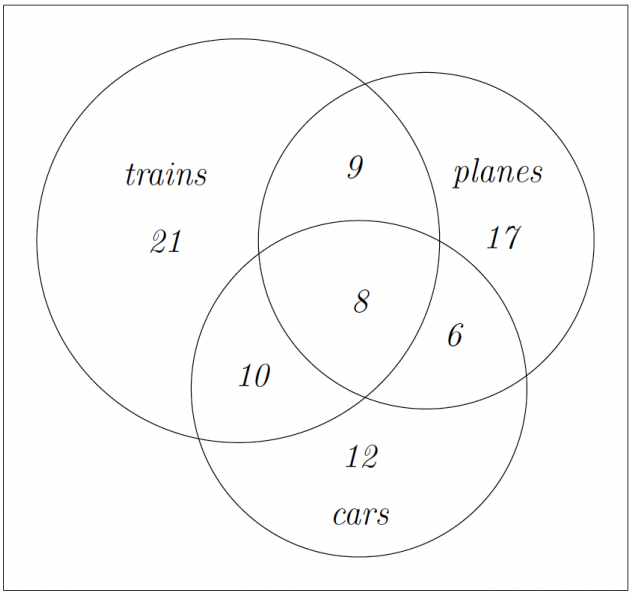
Two sets are called **disjoint** is they have no elements in common. In other words, the sets A and B are disjoint if and only if $A \cap B = \varnothing$.

Problem 18. *Give an example of two disjoint sets.*

Problem 19. *Assume the sets A and B are disjoint. Draw the corresponding Venn diagram.*

Problem 20. *Assume $A \cap B \neq \varnothing$, $B \cap C \neq \varnothing$ and $A \cap C = \varnothing$. Draw the corresponding Venn diagram.*

Problem 21. *Gregory asked 100 kids whether they were collecting die-cast models of cars, trains, and airplanes.*



According to the Venn diagram above, how many kids were collecting

- *planes?*
- *trains and planes?*
- *trains and cars, but not planes?*
- *all of them?*
- *none of them?*

Problem 22. *Prove the* ***Inclusion Exclusion Principle:*** *for any sets* A, B, $|A \cup B| = |A| + |B| - |A \cap B|$.

When dealing with sets, we may define a **universal set** U under consideration of a specific context or problem. All sets we work with are subsets of the universal set. This way, for a set A we define its **complement** ${}^cA$ as the set of all elements in U that aren't in A.

Problem 23. *Let* $U := \{x \in \mathbb{N} : x \geq 1$ *and* $x \leq 10\}$ *and* $A = \{2, 3, 9, 6\}$. *What is* ${}^cA$*?*

Problem 24. *For each expression, draw a Venn diagram and represent it as a shaded area:*

- $A \cup B$
- $A \cap {}^cB$
- ${}^c(A \cup B)$
- ${}^cA \cap (B \cup C)$

Problem 25. *When learning set cardinalities, we have only worked with finite sets. Do you think infinite sets (like* $\mathbb{N}$ *and* $\mathbb{Q}$*) have cardinalities? (No answer here is right or wrong.)*

To answer the last question, yes! Infinite sets have cardinalities that are infinite. But not all infinities are the same size!

To better understand this, we must to find a way to compare two infinities. For nonempty sets A, B, define their **product** as

$$A \times B := \{(a, b) : a \in A; b \in B\}$$

containing **ordered pairs** of elements from A and B (ordered pairs are lists of two elements).

A **function** $f : A \to B$ is a special subset of $A \times B$ that satisfies two conditions:

- $\forall a \in A,\ \exists b \in B$ s.t. $(a, b) \in A \times B$. (*Every input of the function has at least one output.*)
- Given $a \in A$ and $b, b' \in B$, if $(a, b), (a, b') \in A \times B$ then $b = b'$. (*Every input has at most one output.*)

Given $(a, b) \in f$, we can say in standard English that f *maps* x *to* y.

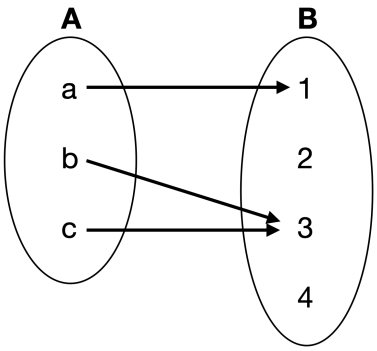
In the previous example, we call A the **domain** of f and B the **codomain** of f. In standard English, these rules say that *every element* a *in the domain* A *corresponds to exactly one (not necessarily unique) element which we call* $f(a)$ *in the codomain* B.

We define the **image** under f of a subset X of A as $f(X) = \{b \in B : \exists x \in X \text{ s.t. } (x, b) \in f\}$.
The image of a function is the image of its domain (this is not necessarily equal to the codomain).

A function is **injective** or **1-to-1** if given $a \in A$ and $b, b' \in B$, if $(a, b), (a, b') \in A \times B$ then $b = b'$.
(*Every input of the function has a unique output.*)

A function is **surjective** or **onto** if $\forall b \in B,\ \exists a \in A$ s.t. $(a, b) \in A \times B$.
(*Every element of* B *is in the image of* f. Equivalently, *the function's image is equal to its codomain.*)

A function that is both 1-to-1 and onto is called **bijective**.
We can visualize a function by showing arrows from elements in the domain to elements in the codomain. For example, if $A = \{a, b, c\}$ and $B = \{1, 2, 3, 4\}$, then the function $f = \{(a, 1), (b, 3), (c, 3)\}$ can be visualized as:



Problem 26. *What is* $f(a)$*? What is* $f(\{b, c\})$*? What is the image of* f*?*

Problem 27. *Is the example function injective? Is it surjective?*

Problem 28. *Can you construct a function $g : A \to B$ that is injective? Can you construct g such that it is surjective? For each case, make a diagram or prove you cannot.*

Problem 29. *Can you construct a function $h : B \to A$ that is injective? Can you construct h such that it is surjective?*

Problem 30. *For finite sets A, B, under what condition can we construct a injective/surjective/bijective function $f : A \to B$?*

Problem 31. *Using our knowledge of functions, how can we compare the cardinalities of finite sets?*

Problem 32. *Can you construct a function $f : \mathbb{N} \to \mathbb{N}$ that is bijective? Can you construct f to be injective but not bijective?*

Problem 33. *Does our logic for finite sets work with infinite sets? If not, how can we compare the cardinalities of infinite sets? (Check with an instructor before moving on.)*

Challenge Potpourri. *Let sets A, B, C be arbitrary, with an arbitrary universal set X.*
For each problem, determine if the statement is necessarily true or not.
(Try not to look at past problems unless you are <u>really</u> stuck.)

Hint: Don't fall for the trick questions, I believe in you!

1. $\{1,2\} = \{2,1\}$ *Necessarily true* ☐ *Not necessarily true* ☐

2. $(1,2) = (2,1)$ *Necessarily true* ☐ *Not necessarily true* ☐

3. $\{\varnothing\} \subseteq A$ *Necessarily true* ☐ *Not necessarily true* ☐

4. $A \subseteq B \Rightarrow A \cup B = B$ *Necessarily true* ☐ *Not necessarily true* ☐

5. $A \not\subseteq B \Rightarrow$ *there exists* $a \in A$ *such that* $a \notin B$ *Necessarily true* ☐ *Not necessarily true* ☐

6. $A \subseteq B \Rightarrow$ *there exists* $b \in B$ *such that* $b \in A$ *Necessarily true* ☐ *Not necessarily true* ☐

7. $A \subseteq B \Rightarrow |A| < |B|$ *Necessarily true* ☐ *Not necessarily true* ☐

8. $\mathcal{P}(\varnothing) = \{\varnothing\}$ *Necessarily true* ☐ *Not necessarily true* ☐

*(The **power set** of A, $\mathcal{P}(A)$, is the set of all subsets of A.)*

9. $\{0\} \in \mathbb{N}$ *Necessarily true* ☐ *Not necessarily true* ☐

10. $A \cap (B \cup C) = (A \cap C) \cup (B \cap C)$ *Necessarily true* ☐ *Not necessarily true* ☐

11. $|A| < |{}^c A|$ *Necessarily true* ☐ *Not necessarily true* ☐

12. $|A| + |B| = |A \cup B| + |A \cap B|$ *Necessarily true* ☐ *Not necessarily true* ☐

13. $A \cup (B \cap C) = (A \cup C) \cap (B \cup C)$ *Necessarily true* ☐ *Not necessarily true* ☐

14. *At least one is true:* $A \cap B \neq \varnothing, A \cap {}^c B \neq \varnothing, A = \varnothing$ *Necessarily true* ☐ *Not necessarily true* ☐

15. $|X| \geq 4 \Rightarrow \left|\{A \subseteq X : |A| = 1\}\right| \leq \left|\{A \subseteq X : |A| = 3\}\right|$ *Necessarily true* ☐ *Not necessarily true* ☐