GAMES

JUNIOR CIRCLE 10/21/2012

1. Warm Up: The Last Word

Yesterday was the Presidential Debate for Math Circle! Our two candidates, Isaac and Derek, had to prove that they would make a great president. The debate will started at 4:00 PM, and it ended at 4:05 PM. Isaac and Derek took turns arguing with eachother. On their turn, each candidate was allowed to argue their point for 1 or 2 minutes, and then their opponent was allowed to argue for 1 or 2 minutes. This continued until the debate was over, a total of 5 minutes later. As we all know, whoever gets the last word in at the debate wins. Here is a transcript of the debate:

Debate Transcript

4:00-4:01 **Isaac**: I will make a better President, because I prefer prime numbers over the Fibonacci numbers. For example, I enjoy $2, 3, 5, 7 \dots$

4:01-4:02 **Derek**: Lies! I tell you, the Fibonacci numbers are better than the prime numbers! I really enjoy the numbers $1, 2, 3, 5, 8, 13, 21 \dots$

4:02-4:03 **Isaac**: May I quickly note that my opponent says that there are not more even numbers than odd numbers! But look at this list of even numbers! $2, 4, 6, 8 \dots$

4:03-4:05 **Derek**: But they are not more numerous than the digits of $\pi : 3.14159265359 \dots$

So last night, Derek won the debate because he got the last word in. Of course, the night could have turned out a differently if Isaac had picked a better debating strategy.

Problem 1. If Isaac and Derek hold a 10 minute debate, and each of them is only allowed to argue for one minute, and Isaac starts the debate, who will win the debate?

Jeff Hicks/Los Angeles Math Circle/UCLA Department of Mathematics .

Problem 2. Jeff and Morgan are not as good of debaters, so when they have their debate it is only 3 minutes long, and their talking points are allowed to be one or two minutes long. If Jeff starts the debate, why will he always lose? Describe Morgan's strategy.

Problem 3. Jonathan is the moderator of the next debate between Isaac and Derek. The Debate will be 6 minutes long, and the talking points can be one or two minutes long. If he wants Isaac to win, who should he have start the debate?

Problem 4. After going to debate camp, Jeff and Morgan now know how to hold longer debates. In order to show off, they decide to have a 25 minute debate. Again, they are allowed to have talking points of either one or two minutes long. If Morgan starts the debate, can Morgan win every time?

2. Math Games

Today we want to look at mathematical games. What are some properties that a math game should have?

- There should be a winner and a loser (no ties, and the game must stop)
- There should be no luck involved
- No secrets!

It turns out that if a game is ”math", then there can be a winning strategy. A winning strategy is a method that makes you win no matter what moves your opponent may make. But how can we find the perfect strategy? Let us look at a simple game.

2.1. **Jonathan's Favorite Numbers.** In this game, we start with 10 rocks in a pile. On your turn, you may remove 1, 3, or 4 stones from the pile (These are Jonathan's Favorite Numbers). The person who takes the last stone wins.

Let's draw a number line to represent the different number of stones that can be left in the game:

$0 \quad 1 \quad 2 \quad 3 \quad 4 \quad 5 \quad 6 \quad 7 \quad 8 \quad 9 \quad 10$

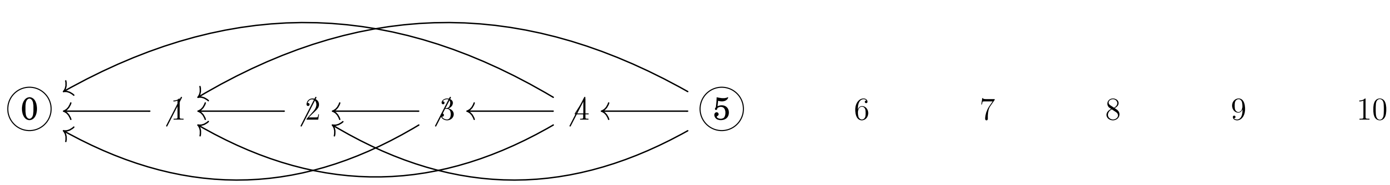
Certainly, if I move into the position where there are 0 stones left, this is a good move, because I win! Let's circle this position.

$\textcircled{0} \quad 1 \quad 2 \quad 3 \quad 4 \quad 5 \quad 6 \quad 7 \quad 8 \quad 9 \quad 10$

Now, If I were to move into the 1 position, would that be a good strategy? No, because then my opponent would move into the 0 position and win. Let's put an X over the 1 to remember that it's not good to move to.

$\textcircled{0} \longleftarrow \not{1} \quad 2 \quad 3 \quad 4 \quad 5 \quad 6 \quad 7 \quad 8 \quad 9 \quad 10$

If we go on and check a few more positions, what do we find?



2.2. **Safe and Unsafe Positions.** Let us look at the possible positions in a game: in this case, the position of the game is given by how many stones remain. We will call a position a **Safe** position (or S position) it is a good idea to move to that square. We will call a position a **Unsafe** position (Or U position) if you move into that position then your opponent has a winning strategy. What are some properties of S and U positions?

- From a S position, you cannot move into another S position. This is because if I move into a safe position I should be able to find a winning strategy. This means that no matter how my opponent moves, they should not be able to find a winning strategy. So they cannot move into a S position.
- From a U position, you must be able to move into a S position. This is because if I move into a unsafe position, it is possible for my opponent to find a winning strategy, which means that they would move into a S position.

Let's return to Jonathan's Favorite Numbers, and classify some positions as safe or unsafe.

Sometimes instead of a number line, it is easier to fill out a table instead. Remember, a position is safe only if there are no other safe positions to move into.

Position	Positions you can Move into	Is this Position Safe?
0	None	S (Because if you move here, you win)
1	⓪	U
2	1	S
3	⓪, ②	U
4	⓪, 1, 3	U
5	1, ②, 4	U
6	②, 4, 5	U
7	3, 4, 6	U
8	4, 5, ⑦	U
9	5, 6, 8	S
10	6, ⑦, ⑨	U

Problem 5 (The Debate Game)**.** In the debate, each player had to talk for 1 or 2 minutes. Mark the moves that are possible on this number line with arrows.

(0) 1 2 3 4 5 6 7 8 9 10

Can you fill in the rest of this table to find a good strategy for the debates? Remember you win the debate if you get the last word in. I've filled in the first 3 rows

Minutes left in debate	Positions you can Move into	Is this Position Safe?
0	None	S (Because you get the last word in)
1	(0)	U
2	(0), 1	U
3	1, 2	S
4		
5		
6		
7		
8		
9		
10		

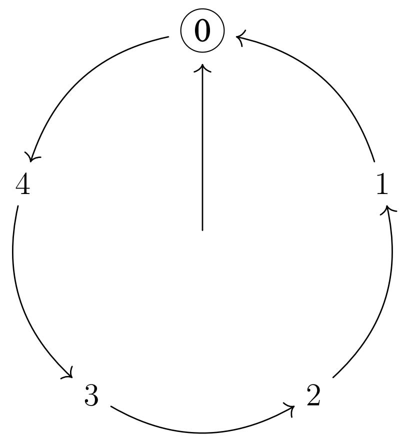
Can you now describe a strategy for the debater if they want to win? If Jonathan starts the debate, how can he always win?

Problem 6 (The Long Debate)**.** In this debate, each player is allowed to take 1 to 4 minutes off of the clock on their turn. Can you fill in the rest of this table to find a good strategy for the debates? Remember you win the debate if you get the last word in. I've filled in the first 3 rows

Minutes left in debate	Positions you can Move into	Is this Position Safe?
0	None	S (Because you get the last word in)
1	(0)	U
2	(0), 1	U
3	(0) 1, 2	U
4		
5		
6		
7		
8		
9		
10		
11		
12		
13		
14		
15		

Can you describe a strategy to help a debater win? If Jeff starts the debate off, is there a strategy that ensures that he wins?

Problem 7 (Morgan's Watch)**.** Morgan owns a watch that he wears to the debates. When the debate starts, the watch points to 0. The debate runs for 15 minutes long, and each contestant must argue for 1 to 4 minutes on their turn.



(a) At the end of the debate, where will the clock be pointing?

(b) If Morgan's opponent makes the first move, can Morgan debate for long enough to bring the minute hand on his watch back to 0? How?

(c) What is special about the 0 position on Morgan's watch?

(d) If Morgan wants to do well at the debate where people are allowed to talk for just 1 or 2 minutes, what kind of watch should he wear? Draw it below!

Problem 8 (The Ultimate Debating Strategy)**.** Let us look a little more at Problem 6.

(a) At what times were all of the S positions at?

(b) Was there a pattern to the S positions?

(c) How would you describe the S positions using $\mod 5$ arithmetic?

(d) Prove that you cannot move from one S position to another, using $\mod 5$ arithmetic (hint: you are subtracting off 1 to 4 minutes with every turn. What does this mean about the S positions?)

(e) If the debate is going to be 132 minutes long, should Morgan start off the debate if he wants to win?

Problem 9 (The Unknown Debate)**.** We will now show how to win all debate games. Suppose that you are allowed to debate for 1 to d minutes.

(a) You win the debate if you move into the position with 0 minutes left. What does that tell you about the position when there is d or less minutes left? What about the position when there is $(d+1)$ minutes left?

(b) Show that if the number of minutes left is congruent to $0 \mod)(d+1)$, that you cannot move to a position where the number of minutes left is again congruent to $0 \mod d+1$

(c) What can you conclude about positions when the number of minutes are left is congruent to $0 \mod d+1$?

3. The Principle of Symmetry

The game of Tabletiles is played on a circular table. On a player's turn, they place a penny on the table. They are not allowed to have two pennies touch in any way. I claim that the first player always wins! How?

Strategy: To beat this game we use the copycat strategy to show that the first player can always win. On the first turn, the first player places a penny in the exact center of the table. Then the second player can place their penny anywhere they would like. The first player keeps on copying the second players move. Why is this a winning strategy?
From a position that is symmetric, it is only possible to move into a position that is not symmetric. This is because adding a single penny breaks the symmetry. Likewise, whenever you copy your neighbor, you are moving from a position that is unsymmetric into one that is symmetric. We now know:

- The winning position is symmetric
- You cannot move from a symmetric position to a symmetric one.
- You can always move from a unsymmetric position to a symmetric one.

Why does this show that they symmetric positions are S positions?

3.1. **Kayles.**

In the game of Kayles, a row of bowling pins are lined up in a row On your turn, you may throw the bowling ball at the pins, knocking down one or two pins that are right next to each other. A player wins when they bowl down the last pin.

○ ○ ○ ○ ○

For instance, the first player might knock down the second and third pin.

○ ○ ○

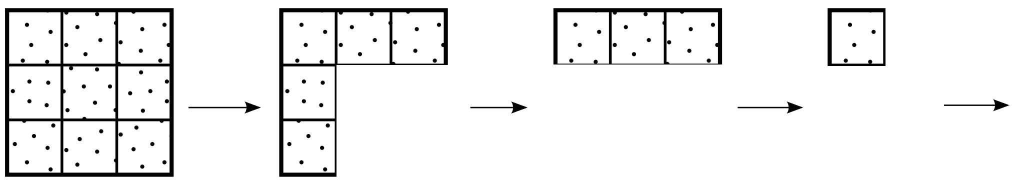
Then the second player might knock down the right two pins

○

Then the first player could knock down the first pin and win.
How can we use the Symmetry strategy to beat this game?

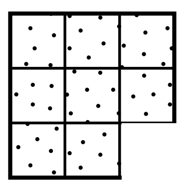
4. The Thief of Strategy

In the game of Chomp, we start with a chocolate bar that measures 3 squares by 3 squares. The top left square is poisoned. On your turn, pick a block that remains in the chocolate bar and then eat every chocolate square that is down and left of it. Obviously, if you eat the poison square, you lose. Here is an example of a set of moves that can happen in the game of Chomp.

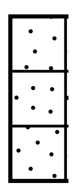


In this example, the first player wins. How can we show that the first player has a winning strategy?

Strategy We will show that if the second player has a winning strategy, then the first player can copy it. If the first player has no winning strategy, then no matter the first move, it is unsafe. Suppose the first player eats just the bottom right square.



Then as this move is unsafe, the second player can now move into a safe position



However, the first player could have moved here on his first move instead of the second player! Therefore, it is always possible for the first player to move into a safe position on his first turn. This is called **strategy stealing**.

5. Jeff's Favorite numbers

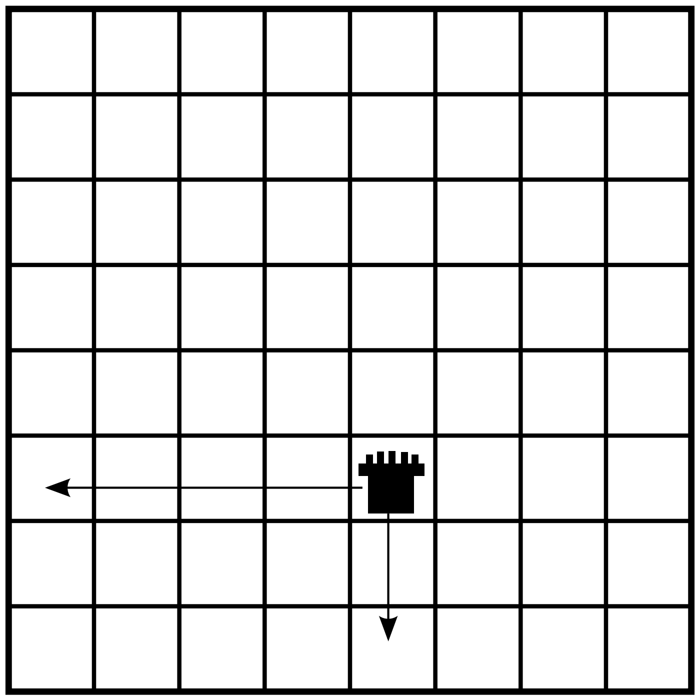
Jeff is fascinated with the powers of two. Whenever he hosts a debates with anybody, he insists that people only debate for a power of two number of minutes.

Minutes left in debate	Positions you can Move into	Is this Position Safe?
0	None	S (Because you get the last word in)
1	(0)	U
2	(0), 1	U
3	1, 2	S
4	(0), (3), 2	U
5	1, (3), 4	U
6	2, 4, 5	
7		
8		
9		
10		
11		
12		
13		
14		
15		

Can you find a pattern? Use modular aritemetic to prove your pattern works?

6. Wyt's Rooks and Queens

In the game of Wyt Rooks, you start with a rook in the top right corner of a chessboard. On your turn, you may move the rook any direction that takes it closer to the bottom left corner.



The person who moves the rook into the corner wins. Why don't we turn this into a game that we can solve with numbers?

Strategy: We want to show that two games are the same. We can think of each square on the chessboard as a pair of numbers, (x, y) where x describes the horizontal position of the square and y describes the vertical distance of the square. For example, in this game the rook is at $(5, 3)$ Let us try to translate our game into this new language. The rook starts at $(8, 8)$, and will end at $(0, 0)$. On a player's turn, they can decrease the x value by moving the rook left, or they can decrease the y value by moving the rook down.
So what are the S positions? When you are in a spot where $x = y$, can you move to a different position with the same horizontal position? You cannot!

Use this information to show that the S positions are those that are on the diagonal of the chessboard.

7. The Nim of Fib

Isaac is moderating the next debate between Morgan and Jonathan. Because Isaac is fair, if Morgan talks for x minutes, he will let Jonathan then talk for as many as $2x$ minutes afterwards. Likewise, if Jonathan talks for y minutes, he will let Mordan talk for as many as$2y$ minutes afterwards.
Isaac starts off the debate at 9 minutes, and let's Morgan start off by talking for as many minutes as he wants, as long as it is less than 9 minutes.

Strategy Here, the game positions are given by two quantites: the amount of time left, and the amount that the previous person talked. Use this chart to figure out the S and U positions in the game: