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1 Variables, equations, and functions

Warm-up

Problem 1.1 *What is the greatest number of parts five straight lines can cut a circle into? Use the blank space below to experiment.*

Answer: *The greatest number of parts is*_____.

1.1 Lesson

Variables and equations

Let us solve two problems from the beginning of the book [1] using variables and equations.

Example 1.1 *Alice and Bob have many coins. Alice has two more coins than Bob. How many coins should Alice give Bob so that they have the same number of coins?*

Soltuion: let Bob have x coins. Then Alice has $x + 2$ coins. Let y be the number of coins Alice should give Bob so that they have the same number of coins. Translating the latter sentence from English into the language of algebra gives one the following equation:

$$x + 2 - y = x + y \tag{1.1}$$

An equation is in a way similar to a seesaw. To keep both in balance, one has to add and subtract the same quantities on both sides. We can

multiply both sides of an equation by the same number, too. We will show in lesson **??** *that division is nothing but multiplication by the reciprocal of the divisor. Hence, one can divide both sides of an equation by a non-zero number without breaking the balance as well.*

Subtracting x from both sides of the equation 1.1 reduces it to the equation $2 - y = y$. Adding y to both sides of the latter equation gives us $2 = 2y$. Dividing both sides by 2 produces the answer $y = 1$. Alice should give Bob one coin so that they have the same number of coins.

The following is problem 1.4 from [1]. Use variables and equations to solve it. Do not guess and match.

Problem 1.2 *Emily and Fabricio collect stamps. Fabricio has one hundred more stamps than Emily. How many stamps should Fabricio give Emily so that they have the same number of stamps?*

Answer: *Fabricio should give Emily* _____ *stamps.*

Let us solve two more problems from the beginning of the book [1] using variables and equations.

Example 1.2 *George, Henry, and Irene collect coins. Henry has one more coin than George. Irene has four more coins than Henry. Irene wants to give the boys some coins so that each of the children has the same number of coins. How many coins should Irene give each boy?*

Solution: let x be the number of coins George has. Then Henry has $x + 1$ coins and Irene has $(x + 1) + 4 = x + 5$ coins. Let y and z be the numbers of coins Irene should give to George and Henry respectively so

that each child has the same number of coins. Then Irene will have $x+5-y-z$ *coins in the end. George will have* $x+y$ *coins, Henry will have* $x+1+z=x+z+1$ *coins, and the latter three numbers will be equal. In particular,* $x+y=x+z+1$*. Hence,* $y=z+1$*. Therefore,* $x+5-y-z=x+5-(z+1)-z=x-2z+4$*. Since the number of coins Irene will have in the end equals the number of coins Henry will have in the end,* $x-2z+4=x+z+1$*. This way,* $-2z+4=z+1$*. The latter boils down to* $3z=3$*, so* $z=1$ *and* $y=2$*.*

For all the children to have the same number of coins, Irene should give two coins to George and one coin to Henry.

The following is problem 2.11 from [1]. Use variables and equations to solve it. Do not guess and match.

Problem 1.3 *Michael, Nick, and Oleg went fishing. Oleg caught seven more fish than Michael. Michael caught five less fish than Nick. How many fish should Oleg and Nick give Michael so that each of the three boys has the same number of fish?*

Answer: *Oleg should give Michael* _____ *fish.*

Nick should give Michael _____ *fish.*

A *function* is a machine for processing some type of objects. If an input cannot be processed, there is no output. If an input can be processed, the machine produces a unique output. The set of all the objects a function can process is called the function's *domain*. The set of all the objects the function produces as outputs is called the function's *range*. Once again, each input from the domain produces a unique output.

Let S be a function taking positive integers as inputs and producing sums of their digits as outputs. For example, $S(2025) = 9$.

Problem 1.4 *Find all the solutions of the equation* $x + S(x) = 2025$.

Answer: _____

Sometimes a function's input consists of more than one object. For example, the heat index reflects how hot a day feels. It depends on the temperature and also on the humidity, the latter showing how much water vapor is there in the air. The greater the humidity, the less sweating helps to cool you down, so the hotter it feels. This way, the heat index, z, depends on the temperature, x, and on the humidity, y. The following table, courtesy of the US National Oceanic and Atmospheric Administration, shows z as a function of x and y.

NOAA national weather service: heat index

Temperature / Relative humidity	80 °F (27 °C)	82 °F (28 °C)	84 °F (29 °C)	86 °F (30 °C)	88 °F (31 °C)	90 °F (32 °C)	92 °F (33 °C)	94 °F (34 °C)	96 °F (36 °C)	98 °F (37 °C)	100 °F (38 °C)	102 °F (39 °C)	104 °F (40 °C)	106 °F (41 °C)	108 °F (42 °C)	110 °F (43 °C)
40%	80 °F (27 °C)	81 °F (27 °C)	83 °F (28 °C)	85 °F (29 °C)	88 °F (31 °C)	91 °F (33 °C)	94 °F (34 °C)	97 °F (36 °C)	101 °F (38 °C)	105 °F (41 °C)	109 °F (43 °C)	114 °F (46 °C)	119 °F (48 °C)	124 °F (51 °C)	130 °F (54 °C)	136 °F (58 °C)
45%	80 °F (27 °C)	82 °F (28 °C)	84 °F (29 °C)	87 °F (31 °C)	89 °F (32 °C)	93 °F (34 °C)	96 °F (36 °C)	100 °F (38 °C)	104 °F (40 °C)	109 °F (43 °C)	114 °F (46 °C)	119 °F (48 °C)	124 °F (51 °C)	130 °F (54 °C)	137 °F (58 °C)	
50%	81 °F (27 °C)	83 °F (28 °C)	85 °F (29 °C)	88 °F (31 °C)	91 °F (33 °C)	95 °F (35 °C)	99 °F (37 °C)	103 °F (39 °C)	108 °F (42 °C)	113 °F (45 °C)	118 °F (48 °C)	124 °F (51 °C)	131 °F (55 °C)	137 °F (58 °C)		
55%	81 °F (27 °C)	84 °F (29 °C)	86 °F (30 °C)	89 °F (32 °C)	93 °F (34 °C)	97 °F (36 °C)	101 °F (38 °C)	106 °F (41 °C)	112 °F (44 °C)	117 °F (47 °C)	124 °F (51 °C)	130 °F (54 °C)	137 °F (58 °C)			
60%	82 °F (28 °C)	84 °F (29 °C)	88 °F (31 °C)	91 °F (33 °C)	95 °F (35 °C)	100 °F (38 °C)	105 °F (41 °C)	110 °F (43 °C)	116 °F (47 °C)	123 °F (51 °C)	129 °F (54 °C)	137 °F (58 °C)				
65%	82 °F (28 °C)	85 °F (29 °C)	89 °F (32 °C)	93 °F (34 °C)	98 °F (37 °C)	103 °F (39 °C)	108 °F (42 °C)	114 °F (46 °C)	121 °F (49 °C)	128 °F (53 °C)	136 °F (58 °C)					
70%	83 °F (28 °C)	86 °F (30 °C)	90 °F (32 °C)	95 °F (35 °C)	100 °F (38 °C)	105 °F (41 °C)	112 °F (44 °C)	119 °F (48 °C)	126 °F (52 °C)	134 °F (57 °C)						
75%	84 °F (29 °C)	88 °F (31 °C)	92 °F (33 °C)	97 °F (36 °C)	103 °F (39 °C)	109 °F (43 °C)	116 °F (47 °C)	124 °F (51 °C)	132 °F (56 °C)							
80%	84 °F (29 °C)	89 °F (32 °C)	94 °F (34 °C)	100 °F (38 °C)	106 °F (41 °C)	113 °F (45 °C)	121 °F (49 °C)	129 °F (54 °C)								
85%	85 °F (29 °C)	90 °F (32 °C)	96 °F (36 °C)	102 °F (39 °C)	110 °F (43 °C)	117 °F (47 °C)	126 °F (52 °C)	135 °F (57 °C)								
90%	86 °F (30 °C)	91 °F (33 °C)	98 °F (37 °C)	105 °F (41 °C)	113 °F (45 °C)	122 °F (50 °C)	131 °F (55 °C)									
95%	86 °F (30 °C)	93 °F (34 °C)	100 °F (38 °C)	108 °F (42 °C)	117 °F (47 °C)	127 °F (53 °C)										
100%	87 °F (31 °C)	95 °F (35 °C)	103 °F (39 °C)	112 °F (44 °C)	121 °F (49 °C)	132 °F (56 °C)										

Key to colors: Caution | Extreme caution | Danger | Extreme danger

Problem 1.5

- *Use the above table to find* $z(82, 90)$.

$z(82, 90) =$_____$°F$

The problem continues to the next page.

- *Which feels more comfortable, (A) the $82°F$ temperature and 90% humidity or (B) the $88°F$ temperature and 40% humidity? Circle the correct answer and explain your reason.*

A B

- *Estimate* $z(90, 82)$.

$z(90, 82) \approx$_____$°F$

The Body Mass Index, or BMI, is a tool that uses a person's weight and height to estimate potential health risks, such as heart diseases and diabetes. BMI is calculated by dividing weight in kilograms by height in meters squared. Adult BMI is categorized as

underweight	normal	overweight	obese
$* < 18.5$	$18.5 \leq * < 25$	$25 \leq * < 30$	$30 \leq *$

Note that the index is quite crude. It doesn't distinguish between muscle and fat, potentially overestimating body fat in muscular people and underestimating it in older people or those with muscle loss.

Problem 1.6 *Let h be a person's height, in meters, and let w be the person's weight, in kilograms. Write down the formula representing the BMI function.*

Answer: $BMI =$ _____

Problem 1.7 *One foot is approximately equal to* 0.3 *of a meter. One pound is approximately equal to* 0.45 *of a kilogram. Write down the formula representing the BMI function where the weight,* x*, is measured in pounds and the height,* y*, is measured in feet so that one could still use the table on page 7.*

Answer: $BMI =$ _____

Problem 1.8 *Fill out the numbers missing in the below table of function values for the function* $z = x + 2y$*. Use the blank space on the right-hand side for work.*

x	4	1	3	
y	2	13		2
z			12	21

The following function

$$|x| = \begin{cases} x & \text{for} \quad x \geq 0 \\ \\ -x & \text{for} \quad x < 0 \end{cases}$$

is called the *absolute value of* x. Geometrically, the function $|x|$ shows the distance of the number x from zero on the number line.

Example 1.3 *Solve the equation* $|x-2|=1$.

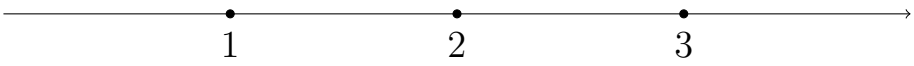
To solve the equation analytically, one has to consider two cases.

Case 1: $x-2\geq 0$ *or equivalently,* $x\geq 2$. *In this case,* $|x-2|=x-2$ *and the equation becomes* $x-2=1$. *The solution is* $x=3$.

Case 2: $x-2<0$ *or equivalently,* $x<2$. *In this case,* $|x-2|=-(x-2)=2-x$. *The equation becomes* $2-x=1$. *The solution is* $x=1$.

Another way to solve the problem is to use geometry of the number line. Solving the equation $|x-2|=1$ *is equivalent to answering the following question.*

What numbers have the distance 1 *from the point* 2*? See the answer on the picture below.*



Problem 1.9 *Find all the solutions of the equation* $|||x-2|-2|-2|=1$.

Answer: _____

Example 1.4 *Find all the integer solutions of the equation*

$$xy + x - 3y = 8.$$

Write down each solution as a pair (x, y) *in parentheses.*

Let us try to factor the left-hand side of the equation:

$$xy + x - 3y = x(y + 1) - 3y = x(y + 1) - 3y - 3 + 3 =$$

$$x(y + 1) - 3(y + 1) + 3 = (x - 3)(y + 1) + 3 = 8.$$

The latter is equivalent to the equation $(x - 3)(y + 1) = 5$. *The only factors the right-hand side has are* ± 1 *and* ± 5. *Hence, we get the following cases:* $x - 3 = 1$, $y + 1 = 5$*;* $x - 3 = 5$, $y + 1 = 1$*;* $x - 3 = -1$, $y + 1 = -5$*; and* $x - 3 = -5$, $y + 1 = -1$. *This way, the answers are:* $(4, 4)$, $(8, 0)$, $(2, -6)$, *and* $(-2, -2)$.

Problem 1.10 *Find all the integer solutions of the equation*

$$xy + 5x - 3y = 2.$$

Write down each solution as a pair (x, y) *in parentheses.*

Answer: _____

Problem 1.11 *A flock of flying cats and three-headed dragons is hovering over the sky of planet Delirium. A person observing the flight has counted 25 heads and 100 legs in all. How many legs has a (healthy) three-headed dragon on the planet?*

Answer: *A healthy dragon has* _____ *legs.*

Problem 1.12 *A rectangle with side lengths 73 and 97 units is tiled with unit squares. How many of these squares does the diagonal of the rectangle cross? Hint: the numbers 73 and 97 do not have common factors.*

Answer: *The diagonal crosses* _____ *squares.*

1.2 Homework

Finish solving all the problems from class. Discuss with parents the problem you liked the most.

Dog years

Dogs grow up quicker than humans do, so people have invented the concept of dog years to try to match dogs of a certain age with the corresponding age in human development. To convert a dog's age in human years into dog years, one must multiply the former number by seven. So, if a dog is two human years old, it is $2 \times 7 = 14$ years old in dog years.

Problem 1.13 *Let x be the age of a dog in human years and let y be the age of the dog in dog years. Write down a formula for y as a function of x.*

Answer: $y =$_____

Problem 1.14 *Molly the dalmatian is 15 human years old. How old is she in dog years?*

Answer: *Molly is*_____ *dog years old.*

Problem 1.15 *Pepper the beagle is two and a half human years old. How old is she in dog years?*

Answer: *Pepper is*_____ *dog years old.*

Problem 1.16 *Eliot the boy is 9 years 11 months old. How old would a dog have to be, if its age in dog years were the same as Eliot's?*

Answer: *The dog would have to be* _____

_____

Problem 1.17 *Let x be the age of a dog in dog years and let y be the age of the dog in human years. Write down a formula for y as a function of x.*

Answer: $y =$_____

Problem 1.18 *Eliot was eight when Parker the dalmatian was born. Will there ever be a time when Parker's age in dog years will be equal to Eliot's age measured in human years? Circle the correct answer and explain your reason.*

Yes *No*

Temperature scales

The Fahrenheit ($^\circ F$) temperature scale is used in the US. Most countries outside of the US use the Celsius ($^\circ C$) scale. Fortunately, it is not hard

to convert from one scale to the other using the following formula. If x is the temperature in $^\circ C$ and y is the same temperature in $^\circ F$, then

$$y = \frac{9}{5}x + 32 \tag{1.2}$$

Problem 1.19 *Convert the following temperatures between Celsius and Fahrenheit. Use the blank space under the table for computations. Round to one decimal digit after the point.*

$x^\circ C$	-10	0	10	20	100			
$y^\circ F$						0	60	100

Problem 1.20 *What temperature is the same number in both Celsius and Fahrenheit scales?*

Answer: *The temperature is* _____.

Problem 1.21 *Formula 1.2 converts Celsius to Fahrenheit. Find the formula for a function converting Fahrenheit to Celsius. As in formula 1.2, use x for Celsius and y for Fahrenheit.*

Answer:_____

Problem 1.22 *Solve the equation* $||x + 1| + 2| = 3$.

Answer: _____

Problem 1.23 *Six Olympic Games competitors have gotten the gold, silver, and bronze medals by scoring 15, 14, and 13 points respectively and 82 points total. How may competitors have gotten each medal? Solve the problem using equations and variables. Do not guess and check.*

Answer: _____

References

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[5] Alexander Shen, *Geometry in Problems*, ISBN 978-1470419219