LAMC Junior Circle **February 24, 2013**

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Warm-up

Problem 1 *What is the smallest natural number such that its double is a perfect square and its triple is a perfect cube?*

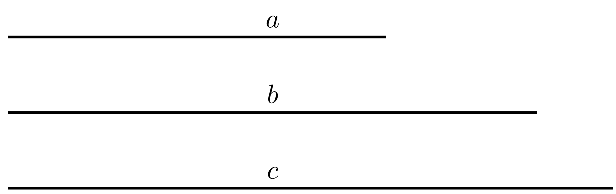
Problem 2 *Can the number* $a^2 + b^2 + c^2$ *be divisible by 5 if neither of the numbers* a*,* b*, and* c *is divisible by 5? Why or why not?*

The number 13 is considered by many as an unlucky number. Equally, Friday is considered by many as an unlucky day. Solve the following problem for a regular, not leap, year.

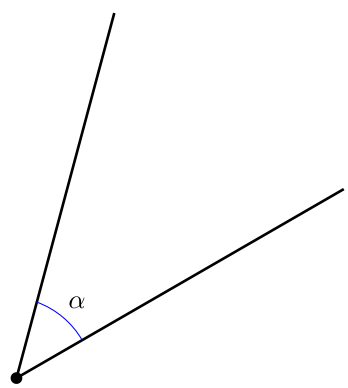
Problem 3 • *Does there exist a year without a Friday the 13th?*

• *What is the maximal number of times Friday the 13th can occur in a year?*

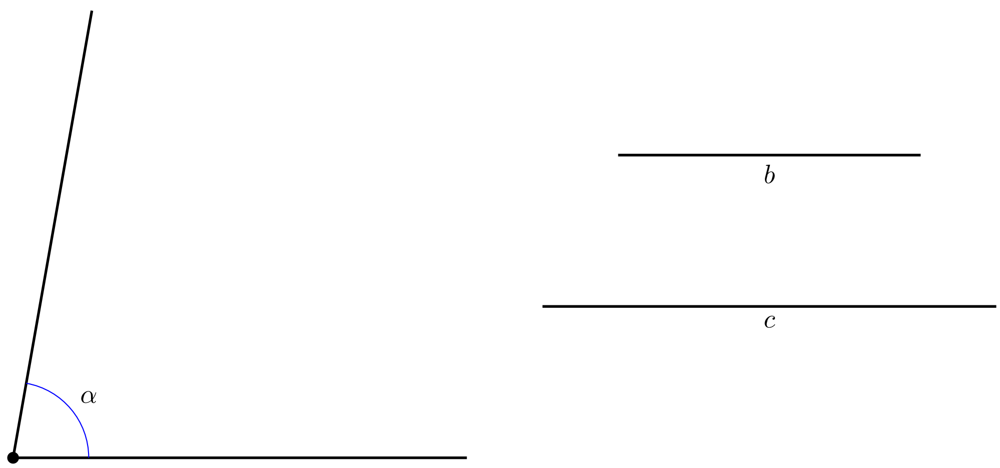
Problem 4 *Using a compass and a ruler, draw a triangle with the given sides a, b, and c in the space below.*



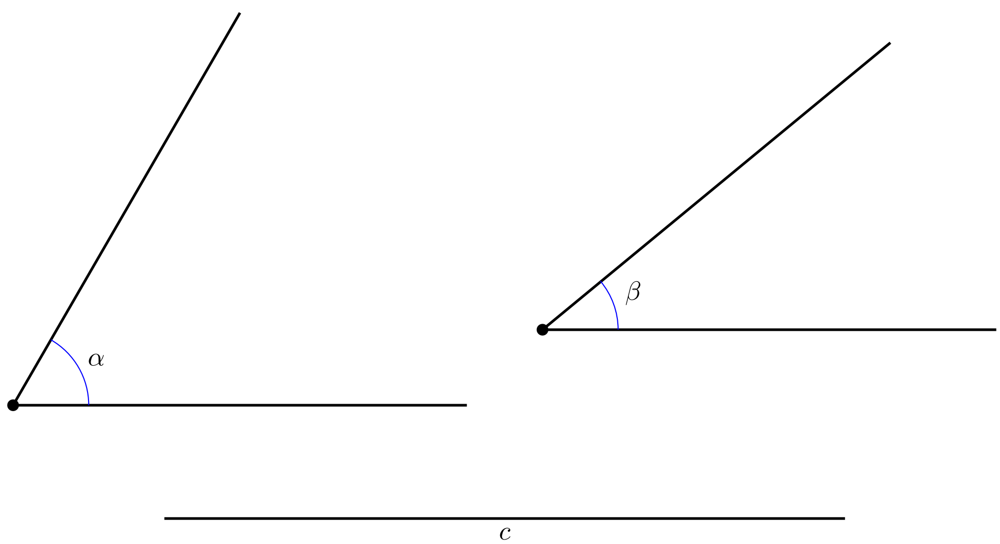
Problem 5 *Using a compass and a ruler, draw the given angle* α *having the given ray as its side.*



Problem 6 *Using a compass and a ruler, draw a triangle with the angle α and adjacent sides b and c given below. In this case, the word "adjacent" means that the vertex of α is an endpoint of the sides b and c.*



Problem 7 *Draw a triangle with the side c and adjacent angles α and β given below. In this case, the word "adjacent" means that the endpoints of c are the vertices of α and β.*



Note that solutions to Problems 4, 6, and 7 (nearly) prove the following theorem fundamental for what follows.

Theorem 1 *Two triangles in the Euclidean plane are equal if either of the following holds.*

- *Their side lengths are pairwise equal.*

$$|a| = |a'|, \quad |b| = |b'|, \quad |c| = |c'|$$

- *They have an angle of equal size, and the lengths of the sides adjacent to the equal angles are pairwise equal.*

$$\alpha = \alpha', \quad |b| = |b'|, \quad |c| = |c'|$$

- *They have a side of equal length, and the adjacent angles are pairwise equal.*

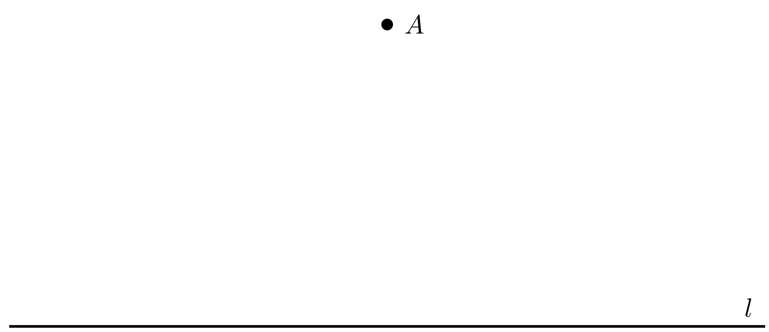
$$|c| = |c'|, \quad \alpha = \alpha', \quad \beta = \beta'$$

Question 1 *What does it mean that two triangles are equal?*

Question 2 *How would you prove the first statement of Theorem 1?*

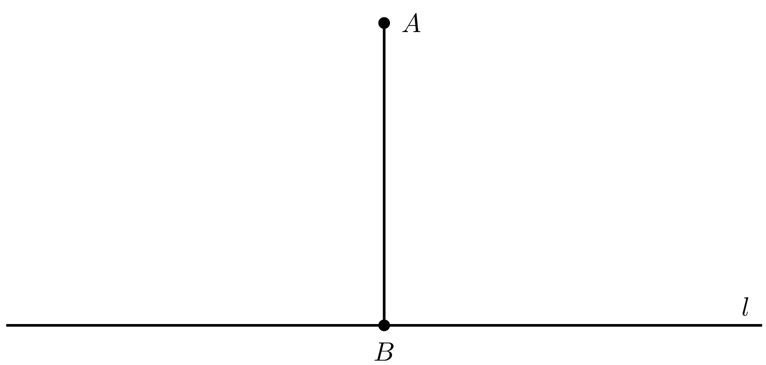
Triangles and circles

Problem 8 *Using a compass and a ruler, construct the shortest possible path from the point A to the straight line l below.*



Use the Pythagoras' Theorem to prove that the path you have constructed is indeed the shortest.

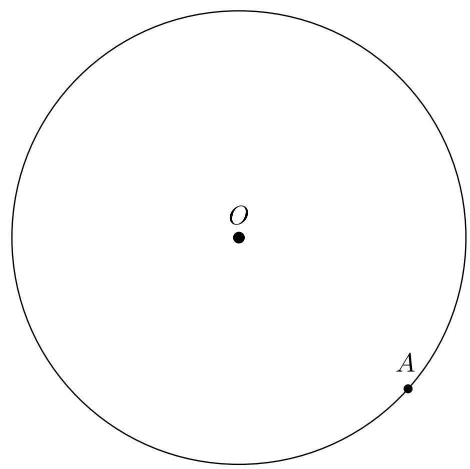
Problem 9 *Let us call B the end of the shortest path opposite to the point A in Problem 8, see the picture below.*



Draw the circle centered at A and passing through B. Can the circle intersect the straight line l at any point other than B? Why or why not?

Definition 1 *A straight line that intersects a circle at one point only is called tangent to the circle at the point.*

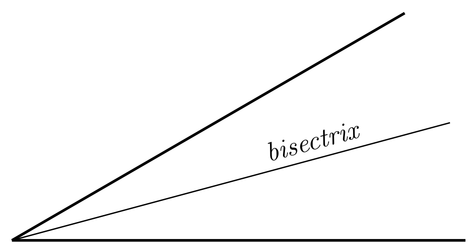
Problem 10 *Use a compass and a ruler to construct a straight line tangent to the below circle at the given point A.*



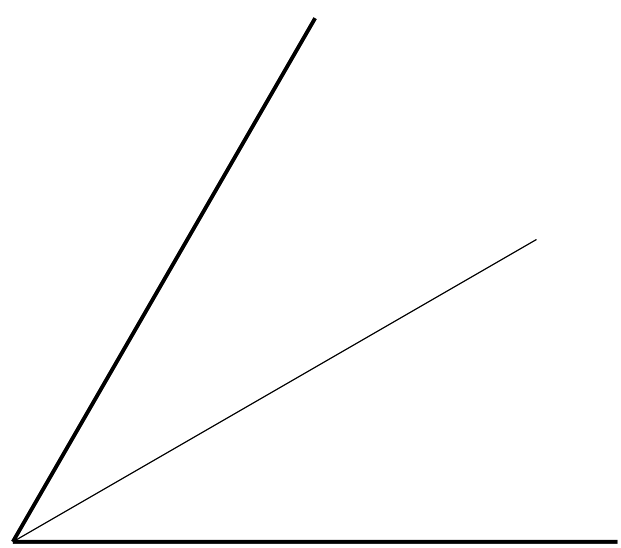
Theorem 2 *A straight line is tangent to a circle if and only if it is perpendicular to the radius drawn to the tangency point.*

Problem 11 *Prove Theorem 2.*

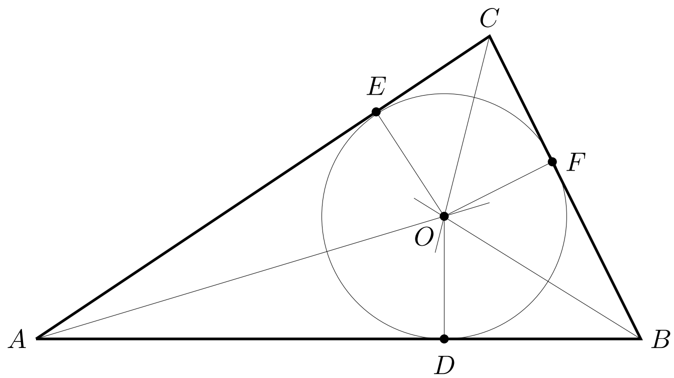
Definition 2 *A bisectrix, or the angle bisector, is a ray that splits the angle into two equal parts.*



Problem 12 *Prove that the points of an angle bisector are equidistant from the sides of the angle.*



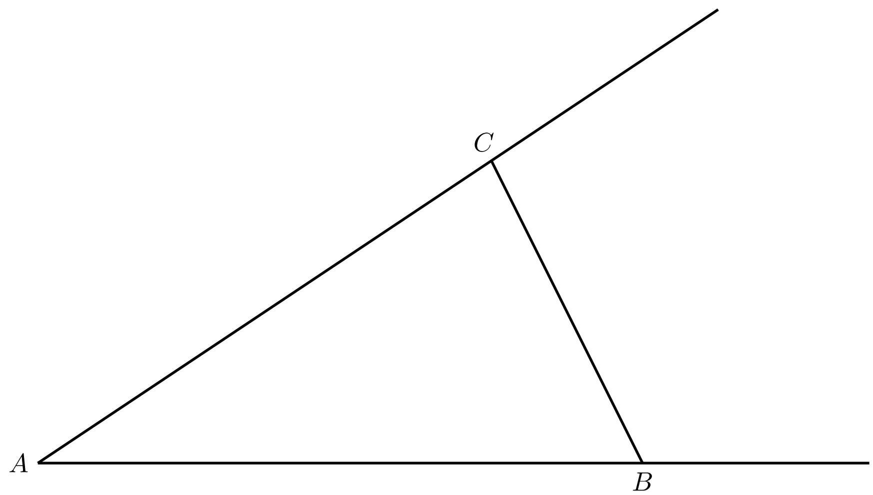
Theorem 3 *Angle bisectors of any triangle in the Euclidean plane have a common point. The point is the center of the circle inscribed in the triangle (tangent to all of its sides).*



Definition 3 *The point is called the incenter of the triangle. The corresponding circle is called the incircle.*

Problem 13 *Prove Theorem 3*

Let us extend two sides of a triangle to rays as below.



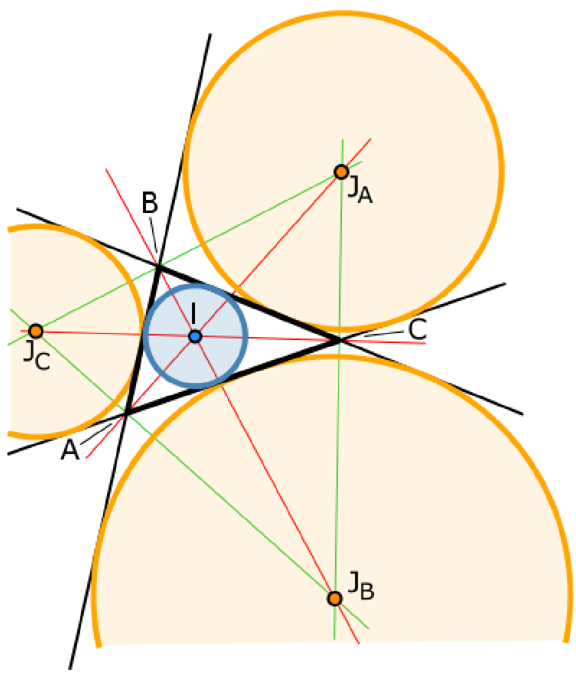
Recall that two angles are called supplementary, if they add up to a straight angle. Let C' and B' be the angles pictured above supplementary to $\angle ACB$ and $\angle ABC$ respectively.

Problem 14 *Prove that the angular bisectors of the angles A, B' and C' intersect at one point.*

The above point is called an *excenter* of the triangle.

Problem 15 *Prove that the excenter constructed in Problem 14 is the center of the circle inscribed in the angles* A*,* B' *and* C'*.*

The circle is called an *excircle*. Below is a picture of a triangle, its incircle and three excircles.

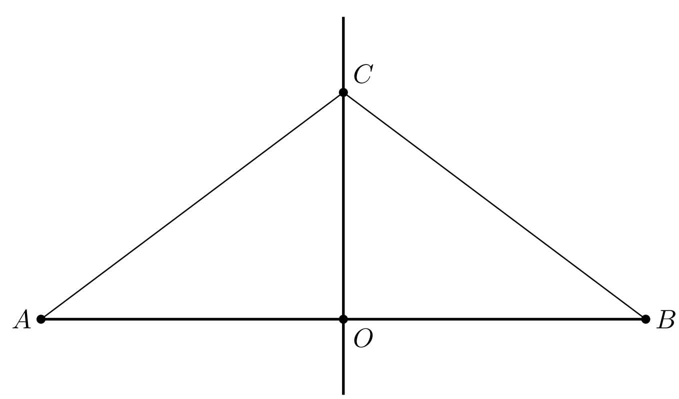


Downloaded form `http://en.wikipedia.org/wiki/Incircle`.

Problem 16 *Consider the triangle $J_A J_B J_C$ on the picture above. Do the vertices of the original triangle ABC indeed lie on its sides as drawn on the picture? Why or why not?*

Definition 4 *A straight line segment bisector is a straight line passing through the middle of the segment. A perpendicular bisector is the bisector forming the right angle with the original line.*

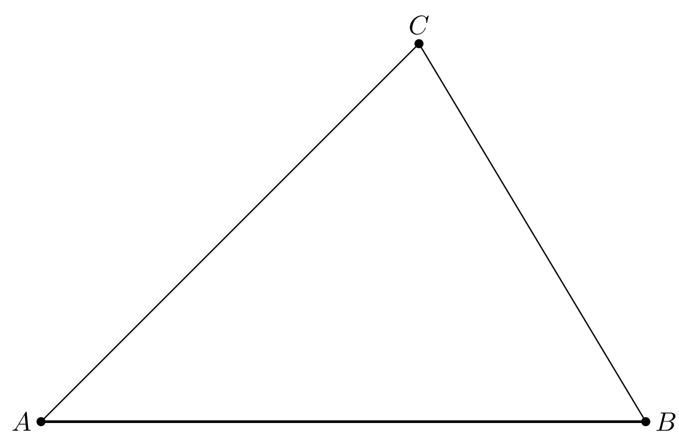
Problem 17 *Prove that the perpendicular segment bisector is the set of all the points in the plane equidistant from the ends of the segment.*



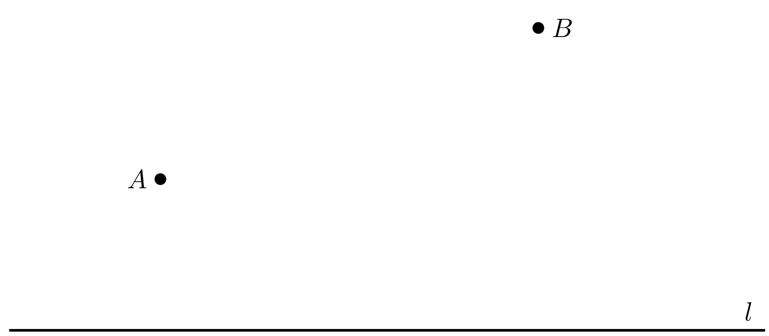
Theorem 4 *Any triangle in the Euclidean plane can be inscribed in a circle.*

Definition 5 *The circle is called the circumcircle of the triangle. The center of the circle is called the circumcenter.*

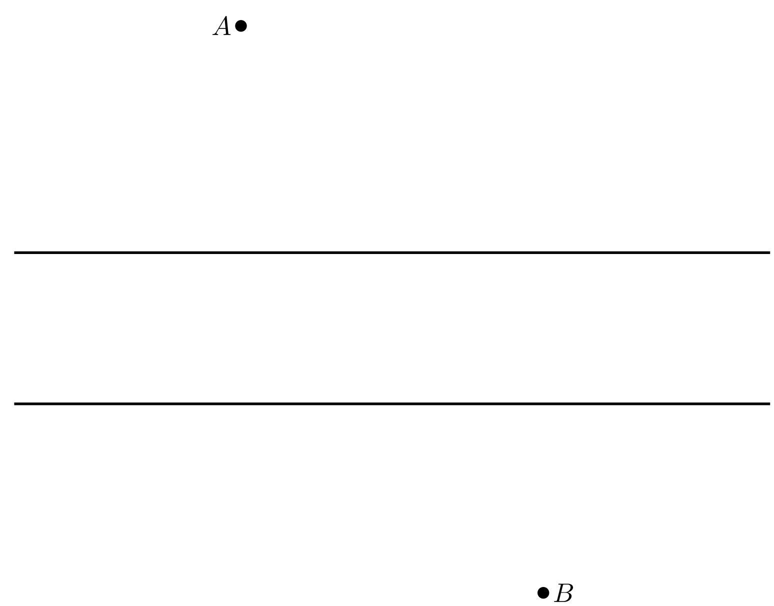
Problem 18 *Prove Theorem 4. Use a compass and a ruler to construct the circumcircle for the triangle below.*



Problem 19 *The cities A and B are located on one side of a straight highway. Use a compass and a ruler to construct the shortest possible road connecting A to the highway and then to B.*

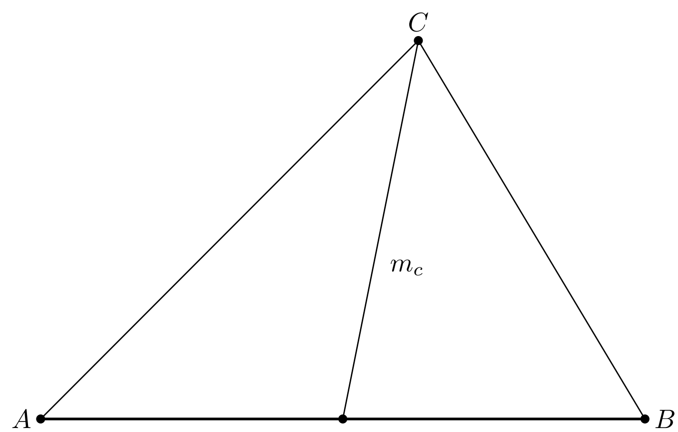


Problem 20 *The cities A and B are separated by a river having parallel straight banks. Using a compass and a ruler, construct the shortest possible highway connecting A to B. The bridge across the river must be perpendicular to the banks.*



Homework

Definition 6 *A segment of a straight line connecting a vertex of a triangle to the middle of the opposite side is called a median.*



Problem 21 *Prove that the medians of a triangle intersect at one point. The point divides each median in the ratio* $2:1$ *counting from the vertex.*

Problem 22 *Solve Problem 3 for a leap year.*