Lines and Planes, Part II

UCLA Math Circle Advanced 1

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For our next topic, we need to introduce some terminology:

Definition 1 *When a plane is divided by a line lying in it, it is divided into 2 parts called* ***half-planes****.*

The line is called the ***edge*** *of each of these half-planes.*

Definition 2 *A figure in space formed by two half-planes (P and Q, Figure 1) which have the same edge (AB) is called a* ***dihedral angle****.*

The line AB is called the ***edge*** *of the dihedral angle.*

The half-planes P and Q are called the ***faces*** *of the dihedral angle.*

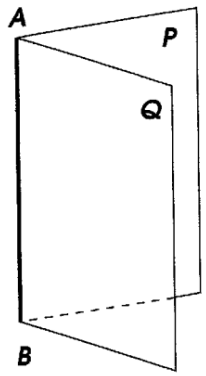

Figure 1

Definition 3 *A dihedral angle divides space into two parts, called the* ***interior*** *(the smaller part bounded by the lesser angle) and the* ***exterior*** *(the larger part bounded by the greater angle).*

If several dihedral angles in a diagram have the same edge, we will denote each of them by four letters, of which the middle two mark the edge and the outer two mark the faces.

Before we focus on proof-related problems, let's do some quick warm-ups on dihedral angles:

Problem 1 *In Figure 2, Locate the dihedral angle $\angle PCDR$. What are its edges and faces? What planes or half-planes lie in its interior and exterior?*

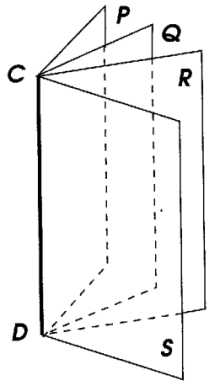

Figure 2

Definition 4 *If from an arbitrary point D on the edge of a dihedral angle, a ray perpendicular to the edge is drawn in each face, then the angle formed by these two rays is called the* ***linear angle*** *of each dihedral angle.*

Problem 2 *In Figure 1, draw geometric structures (points, lines, rays, etc.) to create a* ***linear angle*** *$\angle CDE$ of the dihedral angle $\angle PABQ$.*

Check with an instructor before continuing.

Definition 5 *Two dihedral angles are* ***congruent*** *if they become superimposed when one of them is embedded into the interior of the another, so that their edges coincide.*

Otherwise, the one of them that will lie inside the other is considered ***smaller****.*

Problem 3 *Create 2 different inequalities between the dihedral angles in Figure 2.*

Just like congruency and comparisons, other geometric terms involving 2D angles (vertical, right, complimentary, supplementary) are defined similarly for dihedral angles.

Vertical angles are the two opposite angles at the intersection point of 2 lines.

Theorem 1 *Congruent dihedral angles have congruent linear angles.*

Theorem 2 *The greater of two dihedral angles has the greater linear angle.*

Now back to proofs. We will use the claim-reason proof table like in last class.

Problem 4 *Look at the proof table for Theorem 1, then fill out the table for Theorem 2 (utilize Figure 3).*

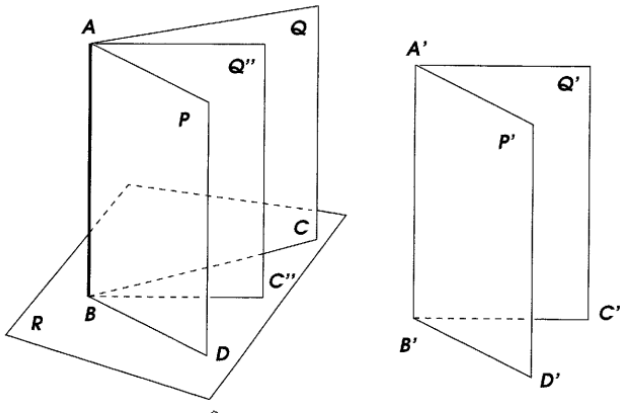

Figure 3

Theorem 1	
Claim	**Reason**
Let $\angle PABQ$ and $\angle P'A'B'Q'$ be two dihedral angles.	Construction
Assume they are congruent.	A given from Theorem 1
Embed $\angle P'A'B'Q'$ into $\angle PABQ$ so that $A'B'$ coincides with AB with $B' = B$, and face P with P'.	Translation and rotation
Then face Q will coincide with face Q'.	Definition of congruence for dihedral angles
Pick point B on the common edge and draw plane $R \perp$ to the edge.	Construction
The intersection of R and the dihedral angles are their linear angles $\angle CBD$ and $\angle C'BD$.	Definition of linear angles
$\angle CBD$ and $\angle C'BD$ are congruent	The angles coincide + definition of congruence for angles

Theorem 2	
Claim	**Reason**
Let $\angle PABQ$ and $\angle P'A'B'Q'$ be two dihedral angles.	
Assume angle $\angle PABQ$ is greater than $\angle P'A'B'Q'$.	
Embed $\angle P'A'B'Q'$ into $\angle PABQ$ so that $A'B'$ coincides with AB with $B' = B$, and face P with P'.	
Then face Q' will occupy a certain position Q'' in the interior of dihedral angle AB.	
Pick point B on the common edge and draw plane $R \perp$ to the edge.	
The intersection of R and the dihedral angles are their linear angles $\angle CBD$ and $\angle C''BD$.	
$\angle CBD$ is greater than $\angle C''BD$	

The converses of these theorems are also true:

Theorem 3 *Congruent linear angles correspond to congruent dihedral angles.*

Theorem 4 *The greater of two linear angles corresponds to the greater dihedral angle.*

Problem 5 *Prove Theorem 3 by making your own claim-reason chart.*

Hint: Use proof by contradiction.
(Start by making a geometric assumption violating Theorem 3, then reach a logical contradiction.)

Definition 6 *If two supplementary angles are congruent to each other, then each of them is called a* ***right dihedral angle****.*

From these theorems, we can reach corollaries 1-5:

Corollary 1 *Right dihedral angles correspond to right linear angles, and vice versa.*

Problem 6 *Prove Corollary 1.*

Corollary 2 *All right dihedral angles are congruent. (Easily shown using 1st corollary)*

Corollary 3 *Vertical dihedral angles are congruent.*

Problem 7 *Prove Corollary 3.*

Corollary 4 *Dihedral angles with respective parallel angles and parallel faces are congruent.*

Problem 8 *What is the 2D equivalent of Corollary 4? (You can just draw a picture of the angles and their relations)*

Corollary 5 *For the measure of a dihedral angle, one takes the degree measure of its linear angle.*

Definition 7 *Two planes are called* ***perpendicular*** *if they intersect and form right dihedral angles.*

Problem 9 *Prove the following:*

If a plane (Q, Figure 4) contains a line (CD) perpendicular to another plane (P), then these two planes are perpendicular.

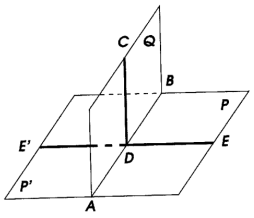

Figure 4

Problem 10 *Prove the following:*

The intersection line (AB, Figure 5) of two planes (P and Q) perpendicular to a third plane (R), then these two planes are perpendicular.

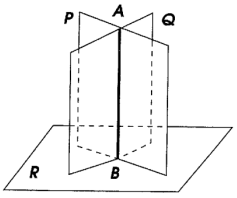

Figure 5

We introduced the idea of projection onto a plane in the previous packet, but now we will give a more general definition of a projection:

(1) The **orthogonal/Cartesian** projection of a point to a given plane is defined as the foot of the perpendicular ray dropped from this point to the plane.

For the sake of brevity we omit the word "orthogonal" and just say "projection."

(2) The **projection of a given figure** (e.g., a curve) to a given plane is defined as the set of projections of all points of this figure.

From these definitions, we can show that the projection of a line onto a plane is also a line. We can do this by proving the following theorem:

Theorem 5 *Given points A, B and a plane P, and letting A', B' be the respective projections of A, B on P, prove that any point M from line AB onto P has its projection M' in line $A'B'$.*

Problem 11 *Draw Theorem 5 below, then prove it.*
Note: Nothing related to dihedral angles is needed for this proof.

Problem 12 *Prove that a plane and a line both making right angles with another line are parallel to each other.*

Problem 13 *Given a plane P and a line $\ell \parallel P$, find a plane containing the line ℓ and making a given angle with the plane P.*

Now let's move on to polyhedral angles:

Definition 8 *Take several angles ASB, BSC, ..., ESF (in Figure 6), which being consecutively adjacent to each other, lie in the same plane around a common vertex S.*

Now, rotate the plane ASB about the side SB so that it forms a dihedral angle with the plane BSC. Now, rotate the plane BSC about the side SC so that it forms a dihedral angle with the plane CSD. Proceed with consecutive rotations. If at the end the first side SA coincides with the last side SF, then the geometric figure formed (in Figure 7) is a ***polyhedral angle****.*

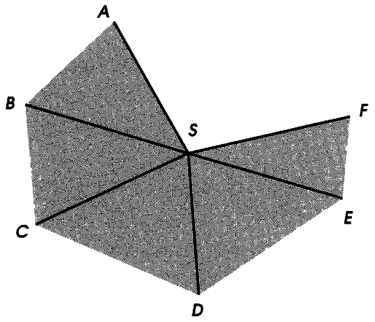

Figure 6

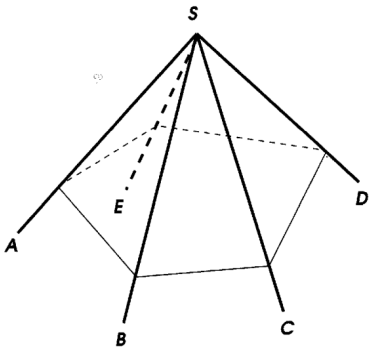

Figure 7

Definition 9 *The angles ASB, BSC, ..., ESA are called the* ***plane angles*** *or* ***faces****,*
their sides SA, SB, ..., SE are called ***edges****,*
and their common vertex S is called the ***vertex*** *of the polyhedral angle.*

Each edge of a polyhedral angle is an edge of a dihedral angle formed from the two adjacent faces. Thus a polyhedral angle has as many edges, or dihedral angles, as it has plane angles.

Definition 10 *The smallest number of faces a polyhedral angle can have is 3, and such angles are* ***trihedral****. Polyhedral angles with 4 faces are* ***tetrahedral****, polyhedral angles with 5 faces are* ***pentahedral****, etc.*

We denote polyhedral angles by a single letter for its vertex (S), or the vertex followed by labels for the edges in their consecutive order ($SABCDE$).

Definition 11 *A polyhedral angle is called* ***convex*** *if it lies entirely on one side of the planes of each of their faces.*

Theorem 6 *In a trihedral angle, each plane angle is smaller than the sum of the other two plane angles.*

Let's aim to prove this theorem for arbitrary trihedral angle $SABC$ (Figure 8). Without loss of generality, let ASC be the greatest of the plane angles. In the interior of the angle, create ray SD in the plane ASC so that the angle ASD as congruent to ASB. Then draw the line AC intersecting SD at point D. Finally, define point E on SB where $|SE| = |SD|$.

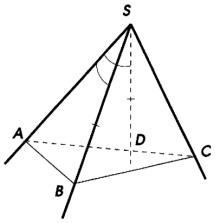

Figure 8

Problem 14 *Create the final part of the proof by showing that:*

$$\angle ASC < \angle ASB + \angle CSB$$

(Hint: Consider the triangular inequality in the triangle $\triangle AEC$.)
(What can we say about the 2 other inequalities?)

Definition 12 *Extend all edges of a polyhedral angle $SABCDE$ (Figure 9) past the vertex S, so they form a new polyhedral angle $SA'B'C'D'E'$ positioned oppositely, which can be called* ***vertical*** *to the original one.*

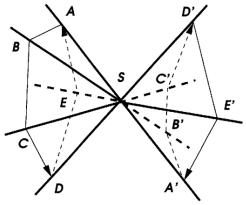

Figure 9

Problem 15 *Looking back to our definition of congruency of two structures by superimposing one on another, are the two polyhedral angles necessarily congruent? Could they still be congruent? If so, when is this the case?*

Such polyhedral angles are often called **symmetric**.

Theorem 7 *Two trihedral angles are congruent if they have a pair of congruent dihedral angles enclosed between two respectively congruent and similarly positioned plane angles.*

Problem 16 *Prove Theorem 7 using Figure 10, where* $\angle ASB \cong \angle A'S'B'$, $\angle ASC \cong \angle A'S'C'$, *and* $BASC \cong B'A'S'C'$.

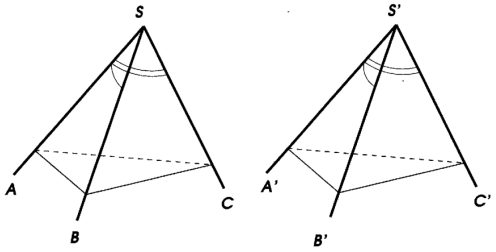

Figure 10

Having covered polyhedral angles, we can now create geometric definitions for polyhedra.

Definition 13 *A* ***polyhedron*** *is a geometric solid bounded by polygons. (plural form is* ***polyhedra****)*

The boundary polygons of a polyhedron are called its ***faces****.*

A common side of two adjacent faces is called an ***edge*** *of a polygon.*

When several faces meet at the common vertex, they form a polyhedral angle, and the vertex of that angle is called a ***vertex*** *of the polyhedron.*

Definition 14 *A straight segment connecting any two vertices, which do not lie in the same face, is called a* ***diagonal*** *of the polyhedron.*

Problem 17 *What is the smallest number of faces a polyhedron can have? what about edges?*

Definition 15 *If a polyhedron lies on one side of the plane of each of its faces, it is* ***convex****.*

From Problem 18 onward, we will only be concerned with convex polyhedra.

Let's look at some different types of polyhedra:

Definition 16 *A* ***prism*** *(in Figure 11) is a polyhedron ($ABCDEA'B'C'D'E'$), two of whose faces ($ABCDE$ and $A'B'C'D'E'$, called the* ***bases*** *of the prism) are congruent polygons with respectively parallel sides, and all other faces ($AA'B'B$, $BB'C'C$, etc., called the* ***lateral edges*** *of the prism) are parallelograms connecting the parallel sides.*

The perpendicular OO' dropped from any point of one base to that of the other is called an ***altitude*** *of the prism.*

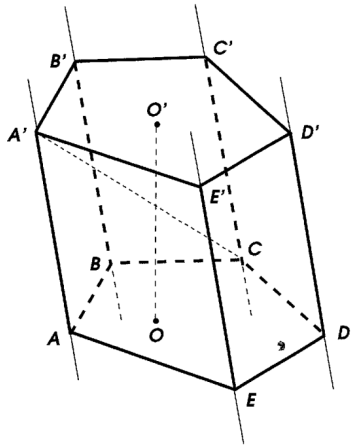

Figure 11

Problem 18 *Fill in the blanks for the following definitions:*

A prism is called ***right*** *if its lateral edges are perpendicular to its bases (and* ***oblique*** *otherwise). Later faces of the right prism are _____.*

A right prism is called ***regular*** *if its bases are regular polygons. Lateral faces of a regular prism are _____.*

A prism is ***triangular*** *if its base is a _____,* ***quadrangular*** *if its base is a _____, etc.*

Definition 17 *A* ***parallelepiped*** *is a prism whose bases are parallelograms. (be careful with its spelling)*

A right parallelepiped is called ***rectangular*** *if its base is a rectangle.*

Three edges connecting to the vertex of a rectangular parallelepiped are called its ***dimensions****, and are separately considered the length, width and height.*

A right parallelepiped with all 3 dimensions congruent is called a ***cube****.*

Theorem 8 *In a parallelepiped:*

1. *opposite faces are congruent and parallel;*
2. *all 4 diagonals intersect at their midpoints.*

Problem 19 *Prove Theorem 8.*

Theorem 9 *In a rectangular parallelepiped, the square of any diagonal is equal to the sum of its dimensions.*

Problem 20 *Prove Theorem 9.*

Definition 18 *A **pyramid** (in Figure 12) is a polyhedron ($SABCDE$), of which one face ($ABCDE$, called its **base**) can be any polygon, and all other faces (called **lateral**) are triangles meeting at a common vertex (S, called the **vertex**).*

*The edges adjacent to the vertex are called the **lateral edges** of the pyramid,*
*and the perpendicular SO dropped from the vertex to the base is called the **altitude**.*

*A plane SBD drawn through the vertex and any diagonal of the base polygon (BD in Figure 12) is called a **diagonal plane** of the pyramid.*

*Pyramids can be **triangular**, **quadrangular**, etc. depending on their bases, similar to prisms. A triangular pyramid is also called a **tetrahedron.***

*A pyramid is **regular** if its base is a regular polygon and the altitude passes through its center.*

*The altitude (SM) of a triangle ($\triangle SCD$) in a pyramid is called an **apothem** (as seen in Figure 12).*

Definition 19 *In Figure 13, the part of the pyramid bounded by the face $ABCDE$ and a section plane $A'B'C'D'E'$ parallel to the base is a polyhedron called a* ***pyramidal frutsum****.*

The faces of the parallel planes are called ***bases****, and the segment OO' of the perpendicular dropped from a point on one base to the other is called the* ***altitude****.*

A frutsum of a regular pyramid is called a ***regular pyramidal frutsum****.*

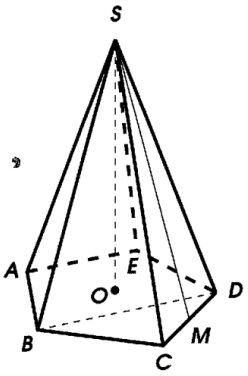

Figure 12

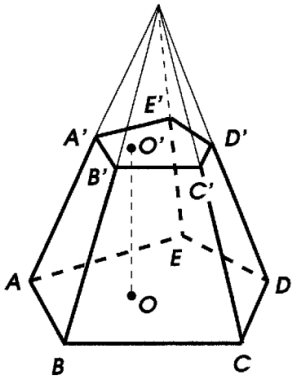

Figure 13

Theorem 10 *If a pyramid is intersected by a plane parallel to the base, then:*

1. *lateral edges and the altitude are divided by this plane into proportional parts;*
2. *the cross section itself is a polygon similar to the base.*
3. *the areas of the cross section and its base are proportional to the squares of the distances from them to the vertex.*

Problem 21 *Prove Theorem 10.*

Definition 20 *The **lateral surface area** of a polyhedron is the total area of all sides of a 3D object excluding the base(s).*

Problem 22 *What is the lateral area of a right prism with a square base and altutude of length h?*

Problem 23 *Write a formula for the lateral area of a right prism, in terms of a few geometric quantities.*

Theorem 11 *The lateral surface area of a prism is equal to the product of a lateral edge and the perimeter of a perpendicular cross section (shown as abcde in Figure 14).*

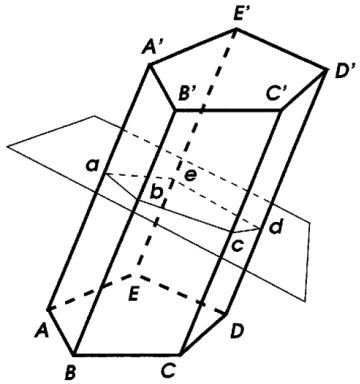

Figure 14

Problem 24 *Prove Theorem 11.*

Theorem 12 *The lateral surface area of a regular pyramid is equal to the product of an apothem and the semiperimeter of the base.*

Problem 25 *Prove Theorem 12.*

Theorem 13 *The lateral surface area of a regular pyramidal frutsum is equal to the product of an apothem and half the sum of the perimeters of the bases.*

Problem 26 *Prove Theorem 13.*

Potpourri (For homework)

Problem 27 *Prove that if the intersection points fo a line with faces of a dihedral angle are the same distance away from an edge, then the angles between the line and the faces are congruent.*

Problem 28 *Prove that, in any convex polyhedral angle, the sum of all the plane angles is less than 360 degrees. Is this also true for all nonconvex polyhedral angles?*

Problem 29 *Show that in a tetrahedron, or a parallelpiped, each face can be chosen for its base.*

Extra-Hard Potpourri (Not for homework, try if you dare!)

Problem 30 *In space, four points A, B, C, D are given such that $AB = AC$ and $DB = DC$. Prove that the lines AD and BC are perpendicular.*

Problem 31 *Prove that a tetrahedral angle can be intersected by a plane in such a way that the cross-section is a parallelogram.*

Problem 32 *Prove that if all diagonals in a prism are* ***concurrent*** *(i.e., pass through the same point), then this prism is a parallelpiped.*