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Lines and Planes

With the concept of points, lines, affine plane, space, and parallel lines explored in our previous packets, let us now dive into some axioms that are the foundation of stereometry.

Axiom 1 *If 2 points of a line lie in a given plane, then every point of the line lies in this plane.*

Axiom 2 *If 2 planes have a common point, then they intersect in a line passing through this point.*

Axiom 3 *Through every 3 non-collinear points, one can draw a plane, and such a plane is unique.*

Question 1 *Prove the following:*

If 2 lines intersect, there is exactly 1 plane containing the lines.

Question 2 *Prove the following:*

Through each line in space, infinitely many planes can be drawn.

Question 3 *Prove the following:*

Prove that if through 3 given points 2 planes can be drawn, then infinitely many planes through these points can be drawn.

Now we will introduce more postulates and theorems for solid geometry.

Axiom 4 *Parallel lines are defined as two non-intersecting lines sharing a plane.*

Axiom 5 *Skew lines (also called non-coplanar lines) are defined as 2 lines that do not share a plane, and they do not intersect.*

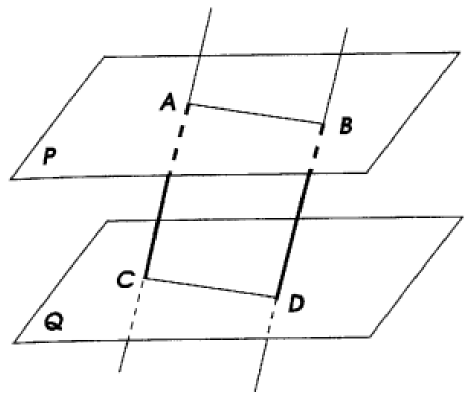
Theorem 1 *If a given line (AB) does not lie in a given plane (P) but is parallel to a line (CD) that lies in it, then the given line is parallel to the plane.*

Question 4 *Draw Theorem 1.*

Theorem 2 *If two intersecting lines (AB and AC) of one plane (P) are respectively parallel to two lines (A'B and A'C") of another plane (P), then these planes are parallel.*

Question 5 *Draw Theorem 2.*

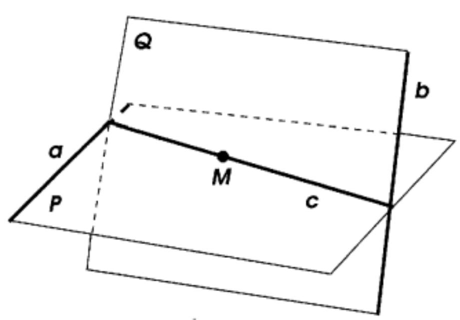
Theorem 3 *The segments (AC and BD), cut off by parallel planes (P and Q) on parallel lines, are of the same length.*



Question 6 *Prove Theorem 3.*

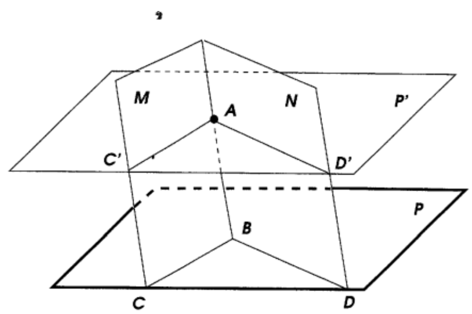
Question 7 *Prove the following.*

Through a given point (M), not lying on either of two given skew lines (a and b), find a line intersecting each of the given ones.



Question 8 *Prove the following.*

Through a given point (A), not lying in a given plane (P), find a plane parallel to the given one.



Question 9 *Prove the following.*

Prove that through every point in space, not lying on a given line, there exists a unique line parallel to the given one.

Question 10 *Prove the following.*

Derive from what we know about parallel lines in a plane: two lines parallel to a third one are parallel to each other. (Key word: equivalence relation.)

Question 11 *Prove the following.*

Prove that for every pair of skew lines a and b, there is a unique pair of planes: one passing through a and parallel to b, the other passing through b and parallel to a, and that these planes are parallel.

Question 12 *Prove the following.*

Can two planes intersect if the first plane contains two lines respectively parallel to two lines contained in the second plane?

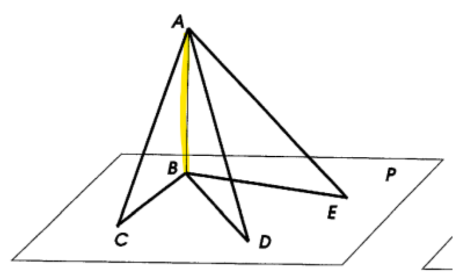
Theorem 4 *If a given line (AO) intersecting a given plane (M) is perpendicular to two lines (OB and OC) drawn in the plane through its intersection point (O) with the given line, then the given line is perpendicular to any other line (OD) drawn in the plane through the same intersection point.*

Question 13 *Imagine if the theorem says it only needs to be perpendicular to 1 line. Can that work?*

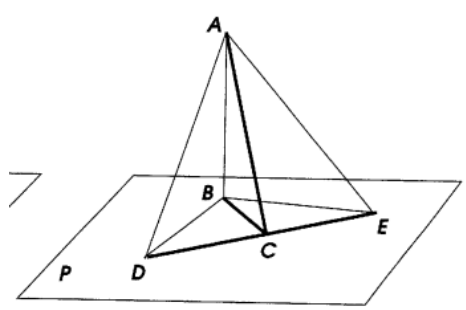
Question 14 *Draw Theorem 4.*

Question 15 *Prove Theorem 4. (Notice the many triangles within the diagram. How can you construct 2 right triangles from an isosceles one?)*

Projection is a very useful tool in stereometry. Here is a brief example: When from the same point A, a perpendicular AB and a slant AC to the same plane P not passing through A are drawn, the segment BC, connecting the feet of the perpendicular and the slant, is called the projection of the slant to the plane P. Thus the segment BC is the projection of the slant AC, the segment BD is the projection of the slant AD, etc.

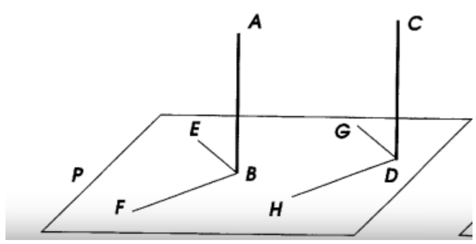


Theorem 5 *If a slant (AC) to a given plane (P) is perpendicular to a line (DE) passing through the foot of the slant and lying in the plane, then the line is perpendicular to the projection (BC) of the slant.*



Question 16 *Prove Theorem 5. (Notice the many triangles; isosceles is useful once more.)*

Theorem 6 *If a plane (P) is perpendicular to one of two given parallel lines (AB), then it is perpendicular to the other (CD).*



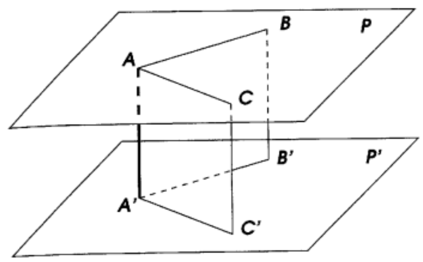
Take for granted that if P is a plane, and we rotated around AB as a an axis. It would appear as a line intersecting another line. We would notice that for 1 line with 2 lines perpendicular to it, the 2 lines perpendicular to each other are parallel to each other, no matter how we rotate.

Question 17 *Prove Theorem 6. (Suppose EB is parallel to DG, and BF is parallel to DH)*

Theorem 7 *If two lines (AB and CD) are perpendicular to the same plane (P), then they are parallel.*

Question 18 *Prove Theorem 7. (Suppose the implication is not true; can it be true?)*

Theorem 8 *If a line (AA') is perpendicular to one of two given parallel planes (P and P'), then it is perpendicular to the other.*



Question 19 *Prove Theorem 8.*

Question 20 *Prove that all points of one of two parallel planes are equidistant from the other.*

Question 21 *Find a line perpendicular to two given skew lines.*

Question 22 *Prove that there are infinitely many lines in space perpendicular to a given line and passing through a given point on it.*