# PROBABILITY

MATH CIRCLE (ADVANCED) 1/27/2013

The likelyhood of something (usually called an **event**) happening is called the **probability.**

**Probability (informal)**: We can calculate probability using a ratio:

$$\frac{\text{want}}{\text{total}}$$

0) Calculate the following probabilities, using the formula

$$\frac{\text{number of ways to get what we want}}{\text{total number of outcomes}}:$$

a) Flipping a fair coin and getting a $H$.

$\frac{1}{2}$

b) Rolling a fair six-sided die and getting a $2$.

$\frac{1}{6}$

c) Randomly picking a number from $1$ to $100$ and getting a number divisible by $3$.

$\frac{33}{100}$

1) Calculate the following probabilities:

a) Flipping a fair coin three times and getting $3H$.

$\frac{1}{8}$

b) Rearranging tiles with the letters of $FORMULA$ and getting a word starting with a vowel.

$\frac{3 \cdot 7!}{8!} = \frac{3}{8}$

c) Flipping a fair coin three times and getting $2H$ and $1T$ (in any order).

$\frac{\binom{3}{1}}{8}$

d) Rearranging tiles with the the letters of $EVENT$ and getting a word starting with $E$.

$\frac{2 \cdot 4!}{5!} = \frac{2}{5}$

Note how we need to be somewhat careful with how we define the total number of outcomes in b) and d). We will return to this later.

**Basic Set Theory**

A **set** just is an unordered collection of objects. We will write sets using $\{\}$, i.e. the set containing $2$ and $4$ is $\{2,4\} = \{4,2\}$.

If $A$ is a set, we use $|A|$ to denote the size of $A$.

In probability, an important set is $\Omega$, called the sample space. $\Omega$ is the collection of all possible outcomes of an experiment.

Suppose $A$ and $B$ are subsets of $\Omega$ (written $A, B \subseteq \Omega$). There are three important set operations:

**Complement**: The complement of $A$ (written $A^c$) is everything in $\Omega$ that is NOT in $A$.

**Union**: The union of $A$ and $B$ (written $A \cup B$) is everything in $A$ OR in $B$ (or in both).

**Intersection**: The intersection of $A$ and $B$ (written $A \cap B$) is everything in $A$ AND in $B$.

2) Suppose $\Omega = \{1,2,3,4,5,6,7,8\}, A = \{1,3,5,6,7\}, B = \{3,5,7\}, C = \{2,4,6\}$. What is:

a) $A \cup C$

$\{1,2,3,4,5,6,7\}$

b) $A \cap B$

$\{3,5,7\}$

c) $A \cap B \cap C$

$\emptyset$

d) $B^c$

$\{1,2,4,6,8\}$

e) $B^c \cap C^c$

$\{1,8\}$

f) $A^c \cup (B \cup C)^c$

$\{1,2,4,8\}$

g) $|A|$

5

h) $|A \cup C|$

7

**Probability**: Suppose $\Omega$ is a finite sample space and every outcome in $\Omega$ is equally likely. If $A \subseteq \Omega$ then

$$\text{the probability of } A = P(A) = \frac{\text{want}}{\text{total}} = \frac{|A|}{|\Omega|}.$$

3) Suppose you roll $2$ die. Let $A$ be the event that the first die is a $6$, $B$ be the event that the sum of the die is $7$, and $C$ be the event that the sum is even.

a) Write out a finite sample space $\Omega$ so that every outcome is equally likely.

$\{(i,j) | 1 \leq i,j \leq 6\}$

b) Calculate $P(A)$

$\frac{6}{36} = \frac{1}{6}$

c) Calculate $P(B)$

$\frac{6}{36} = \frac{1}{6}$

d) Write out the meaning of $A \cap B$. Find $P(A \cap B)$.

$A \cap B$ is the event $\{(6,1)\}$ so it has probability $\frac{1}{36}$

e) Write out the meaning of $B \cap C^c$. Find $P(B \cap C^c)$.

$B \cap C^c = B$ so the probability is $\frac{1}{6}$

4) A poker hand is $5$ cards (in no particular order) dealt from a standard deck of $52$ cards (with $13$ ranks: $2, 3, \ldots 10, J, Q, K, A$ and $4$ suits: hearts, diamonds, clubs, and spades). Find the probability of:

a) a flush (that is, all cards of the same suit).

$$\frac{\binom{4}{1}\binom{13}{5}}{\binom{52}{5}}$$

b) a straight (that is, $5$ ranks in a row, so $A, 2, 3, 4, 5$ or $2, 3, 4, 5, 6$ up to $10, J, Q, K, A$).

$$\frac{\binom{10}{1}\binom{4}{1}^5}{\binom{52}{5}}$$

c) a full house (that is, $3$ of one rank and $2$ or another).

$$\frac{\binom{13}{1}\binom{12}{1}\binom{4}{3}\binom{4}{2}}{\binom{52}{5}}$$

d) exactly two pair.

$$\frac{\binom{13}{2}\binom{11}{1}\binom{4}{2}^2\binom{4}{1}}{\binom{52}{5}}$$

e) exactly one pair.

$$\frac{\binom{13}{1}\binom{12}{3}\binom{4}{2}\binom{4}{1}^3}{\binom{52}{5}}$$

**Axioms of Probability**: Suppose $\Omega$ is a sample space and $A, B \subseteq \Omega$ be an events.

1: $P(A) \geq 0$.

2: $P(\Omega) = 1$.

3: If $A, B$ are **disjoint** ($A \cap B = \emptyset$), then $P(A \cup B) = P(A) + P(B)$.

Note that the following properties follow from the axioms:

4: $P(\emptyset) = 0$.

5: $0 \leq P(A) \leq 1$.

6: $P(A^c) = 1 - P(A)$.

7: $P(A \cup B) = P(A) + P(B) - P(A \cap B)$.

5) Recall $\Omega$ from problem 2). Suppose

$$P(\{1\}) = P(\{3\}) = P(\{5\}) = .15, P(\{2\}) = P(\{4\}) = .05, P(\{6\}) = .25, P(\{7\}) = P(\{8\}).$$

a) Find $P(\{7\})$ and $P(\{8\})$.

$P(\{7\}) = P(\{8\}) = \dfrac{.2}{2} = .1$

b) Find $P(A)$, $P(B)$, and $P(C)$.

$P(A) = .8, P(B) = .4, P(C) = .35$

c) Find the probabilities of the events in a)-f) of 2).

Just add up each part of the event similar to b).

6) Roll three die.

a) Whats the probability of zero $1$'s?

$\dfrac{5^3}{6^3}$

b) Whats the probability of more than zero $1$'s?

$1 - \dfrac{5^3}{6^3}$

c) Whats the probability that the sum is less than $17$?

$1 - \dfrac{3+1}{6^3}$

7) Suppose $P(A) = .4$, $P(B) = .4$, and $P(A \cup B^c) = .7$.

a) Find $P(A \cap B)$.

$.1$

b) Find $P(A^c \cap B^c)$.

$.3$

**Additional Problems**:

8) Prove properties 4-7 of probability using the axioms.

We prove 6 first:

$1 = P(\Omega) = P(A \cup A^c) = P(A) + P(A^c)$ (using 2 and then 3), hence $P(A^c) = 1 - P(A)$.

Then 4 is clear:

$P(\emptyset) = 1 - P(\Omega) = 1 - 1 = 0$ (using 6 and then 4).

5 is clear from 1 and 6.

For 7, note $P(A \cup B) = P(A \cup (B \cap A^c)) = P(A) + P(B \cap A^c)$ (using 3). Thus, it is enough to prove that $P(B) - P(A \cap B) = P(B \cap A^c)$. However, this follows from the fact that $P(B) = P(A \cap B) + P(A^c \cap B)$ (using 3 again).

9) (Simple version of the Birthday Problem)

Suppose that birthdays are equally distributed between all months.

a) Find the probability that in a group of $n$ people, two share the same birthday month.

The probability is 1-$P$(all different birthdays). This is equal to $1 - \dfrac{12!}{(12-n)! \cdot 12^n}$.

b) Find $n$ so that this probability is $\geq 50\%$.

$n = 4$ is roughly $43\%$ while $n = 5$ is roughly $62\%$.

10) (Random Walk in One Dimension)

Suppose you start at position $0$. Each step you take is either $+1$ or $-1$, each with equal probability.

For $k = 2, 4, 6, 8$, find the probability that you never get back to position $0$.

11) (Random Walk in Two Dimensions)

Suppose you start at position $(0, 0)$. Each step you take is one of $+(1, 0), +(0, 1), -(1, 0), -(0, 1)$, each with equal probability.

For $k = 2, 4, 6, 8$, find the probability that you never get back to position $(0, 0)$.