PROBABILITY

MATH CIRCLE (ADVANCED) 1/27/2013

The likelyhood of something (usually called an **event**) happening is called the **probability.**

Probability (informal): We can calculate probability using a ratio:

$$\frac{\text{want}}{\text{total}}$$

0) Calculate the following probabilities, using the formula

$$\frac{\text{number of ways to get what we want}}{\text{total number of outcomes}}:$$

a) Flipping a fair coin and getting a H.

b) Rolling a fair six-sided die and getting a 2.

c) Randomly picking a number from 1 to 100 and getting a number divisible by 3.

1) Calculate the following probabilities:

a) Flipping a fair coin three times and getting $3H$.

b) Rearranging tiles with the letters of $FORMULA$ and getting a word starting with a vowel.

c) Flipping a fair coin three times and getting $2H$ and $1T$ (in any order).

d) Rearranging tiles with the the letters of $EVENT$ and getting a word starting with E.

Note how we need to be somewhat careful with how we define the total number of outcomes in b) and d). We will return to this later.

Basic Set Theory

A **set** just is an unordered collection of objects. We will write sets using $\{\}$, i.e. the set containing 2 and 4 is $\{2, 4\} = \{4, 2\}$.

If A is a set, we use $|A|$ to denote the size of A.

In probability, an important set is Ω, called the sample space. Ω is the collection of all possible outcomes of an experiment.

Suppose A and B are subsets of Ω (written $A, B \subseteq \Omega$). There are three important set operations:

Complement: The complement of A (written A^c) is everything in Ω that is NOT in A.

Union: The union of A and B (written $A \cup B$) is everything in A OR in B (or in both).

Intersection: The intersection of A and B (written $A \cap B$) is everything in A AND in B.

2) Suppose $\Omega = \{1, 2, 3, 4, 5, 6, 7, 8\}, A = \{1, 3, 5, 6, 7\}, B = \{3, 5, 7\}, C = \{2, 4, 6\}$. What is:

a) $A \cup C$

b) $A \cap B$

c) $A \cap B \cap C$

d) B^c

e) $B^c \cap C^c$

f) $A^c \cup (B \cup C)^c$

g) $|A|$

h) $|A \cup C|$

Probability: Suppose Ω is a finite sample space and every outcome in Ω is equally likely. If $A \subseteq \Omega$ then

$$\text{the probability of } A = P(A) = \frac{\text{want}}{\text{total}} = \frac{|A|}{|\Omega|}.$$

3) Suppose you roll 2 die. Let A be the event that the first die is a 6, B be the event that the sum of the die is 7, and C be the event that the sum is even.

a) Write out a finite sample space Ω so that every outcome is equally likely.

b) Calculate $P(A)$

c) Calculate $P(B)$.

d) Write out the meaning of $A \cap B$. Find $P(A \cap B)$.

e) Write out the meaning of $B \cap C^c$. Find $P(B \cap C^c)$.

4) A poker hand is 5 cards (in no particular order) dealt from a standard deck of 52 cards (with 13 ranks: $2, 3, \ldots 10, J, Q, K, A$ and 4 suits: hearts, diamonds, clubs, and spades). Find the probability of:

a) a flush (that is, all cards of the same suit).

b) a straight (that is, 5 ranks in a row, so $A, 2, 3, 4, 5$ or $2, 3, 4, 5, 6$ up to $10, J, Q, K, A$).

c) a full house (that is, 3 of one rank and 2 or another).

d) exactly two pair.

e) exactly one pair.

Axioms of Probability: Suppose Ω is a sample space and $A, B \subseteq \Omega$ be an events.

1: $P(A) \geq 0$.

2: $P(\Omega) = 1$.

3: If A, B are **disjoint** ($A \cap B = \emptyset$), then $P(A \cup B) = P(A) + P(B)$.

Note that the following properties follow from the axioms:

4: $P(\emptyset) = 0$.

5: $0 \leq P(A) \leq 1$.

6: $P(A^c) = 1 - P(A)$.

7: $P(A \cup B) = P(A) + P(B) - P(A \cap B)$.

5) Recall Ω from problem 2). Suppose

$$P(\{1\}) = P(\{3\}) = P(\{5\}) = .15, P(\{2\}) = P(\{4\}) = .05, P(\{6\}) = .25, P(\{7\}) = P(\{8\}).$$

a) Find $P(\{7\})$ and $P(\{8\})$.

b) Find $P(A)$, $P(B)$, and $P(C)$.

c) Find the probabilities of the events in a)-f) of 2).

6) Roll three die.

a) Whats the probability of zero 1's?

b) Whats the probability of more than zero 1's?

c) Whats the probability that the sum is less than 17?

7) Suppose $P(A) = .4$, $P(B) = .4$, and $P(A \cup B^c) = .7$.

a) Find $P(A \cap B)$.

b) Find $P(A^c \cap B^c)$.

Additional Problems:

8) Prove properties 4-7 of probability using the axioms.

9) (Simple version of the Birthday Problem)

Suppose that birthdays are equally distributed between all months.

a) Find the probability that in a group of n people, two share the same birthday month.

b) Find n so that this probability is $\geq 50\%$.

10) (Random Walk in One Dimension)

Suppose you start at position 0. Each step you take is either $+1$ or -1, each with equal probability.

For $k = 2, 4, 6, 8$, find the probability that you never get back to position 0.

11) (Random Walk in Two Dimensions)

Suppose you start at position $(0, 0)$. Each step you take is one of $+(1, 0), +(0, 1), -(1, 0), -(0, 1)$, each with equal probability.

For $k = 2, 4, 6, 8$, find the probability that you never get back to position $(0, 0)$.