1 Double ratio.

Problem 1: Double ratio as invariant of fractional transforms. Given 4 distinct points on the projective line (a, b, c, d) we define their double ratio via

$$(a, b; c, d) = \frac{c-a}{b-c} \Big/ \frac{d-a}{b-d}. \tag{1}$$

(a) Convince yourself that the above formula makes sense even if one of the points is ∞.

Solution:

(b) Find a formula equivalent to (1) in terms of projective coordinates $a = [a_1 : a_2], b = [b_1 : b_2], c = [c_1 : c_2], d = [d_1 : d_2]$. Notice that the formula that you found gives an alternative proof of a).

Solution:

(c) Recall that a fractional linear transform is given by the formula

$$F_{\begin{bmatrix} a & b \\ c & d \end{bmatrix}}([z_1 : z_2]) = [az_1 + bz_2 : cz_1 + dz_2], \quad ad - bc \neq 0.$$

We also recall that fractional linear transforms form a group, and the inverse of F_A is given by $F_{A^{-1}}$, where A is a matrix as above.

Verify that a fractional linear transform F which sends the triple (x_1, x_2, x_3) with $x_1, x_2, x_3 \in \mathbb{P}^1 \setminus \{\infty\}$ to $(0, 1, \infty)$ can be given by the formula

$$F(x) = (x, x_2; x_1, x_3).$$

Solution:

(d) Verify that F as above is unique. In particular, it follows that any triple of points can be mapped to any triple of points, and such a fractional linear transform is unique.

Solution:

(e) Show that $\lambda = (\lambda, 1; 0, \infty)$.

Solution:

(f) Check that $(F(x_1), F(x_2); F(x_3), F(x_4)) = (x_1, x_2; x_3, x_4)$ for an arbitrary fractional linear transform F.

Solution:

(g) Use all the above to conclude that any two tuples of 4 points on $\mathbb{P}^1$ are related by a fractional linear transform if and only if their double ratios are equal. This means that the double ratio is a **complete invariant** of quadruples of points on the projective line.

Solution:

Problem 2: Double ratio on a plane. a) Let $(\ell_1, \ell_2, \ell_3, \ell_4)$ be a quadruple of 4 lines passing through a common point P. We define

$$(\ell_1, \ell_2; \ell_3, \ell_4) = \frac{\sin(\angle \ell_3 \ell_1)}{\sin(\angle \ell_2 \ell_3)} \Big/ \frac{\sin(\angle \ell_4 \ell_1)}{\sin(\angle \ell_2 \ell_4)}. \tag{2}$$

Let (A, B, C, D) be a quadruple of points on a line $\ell \subset \mathbb{P}^2$. Show that

$$(PA, PB; PC, PD) = (A, B; C, D),$$

where $P \notin \ell$ a point on the projective plane $\mathbb{P}^2$. Make sure to work out the case when one of the points (A, B, C, D) is at infinity.

Hint: recall the formula of the area of a triangle in terms of sin *of one of its angles.*

Solution:

Definition 1. We say that a quadruple of points/ lines is **harmonic** if its double ratio is equal to -1.

b) Show that $(A, B; M, \infty) = -1$, where M is a midpoint of the segment AB (here all points are on one projective line).

Solution:

c) Let internal and external bisectors of angle C in the triangle ΔABC intersect the line AB at points X and Y. Show that

$$(A, B; X, Y) = -1.$$

Hint: use a) and b) instead of direct computation.

Solution:

(d) Let X be a point on a segment AB. Construct a point Y on the line AB such that $(A, B; X, Y) = -1$.

Solution:

2 Projective transformations of a plane.

Problem 3. (a) Generally, a **projective transformation** $F_L \colon \mathbb{P}(V) \to \mathbb{P}(V)$, where V is a vector space of $\dim = n + 1$ is given by a map induced on $\mathbb{P}(V)$ by an arbitrary linear transformation of *full rank* (a multiplication by a matrix of *non-zero determinant* in some basis)

$$L \colon V \to V.$$

- Check that a linear transformation indeed induces a well-defined map on $\mathbb{P}(V)$.
- Check that the full-rank assumption implies that the transformation is invertible.

Hint: do not use coordinates, you only need basic properties of linear transformations (ask us if you do not know them, this is completely fine!).

Notice that for $n = 1$ we associated a 2×2-matrix to describe a fractional linear transform, so a projective transformation is a natural generalization of a fractional linear transform.

Solution:

(b) In case $n = 2$, consider L given by

$$A_L = \begin{pmatrix} d & b_1 & b_2 \\ c_1 & a_{11} & a_{12} \\ c_2 & a_{21} & a_{22} \end{pmatrix}.$$

Show that the image of $\mathbf{v} = (x, y) \in \mathbb{R}^2$ can be computed via

$$v \mapsto \frac{1}{\mathbf{b} \cdot \mathbf{v} + d}(A\mathbf{v} + \mathbf{c}),$$

where A is a 2×2-matrix, and $\mathbf{b}, \mathbf{c} \in \mathbb{R}^2$.

Solution:

Problem 4: Perspective projection. The goal of this problem is two show that for a point $P \in \mathbb{P}^3$, and two projective planes $\pi, \pi' \subset \mathbb{P}^3$ the projection from π' to π given by sending $A \in \pi'$ to $F(A) = PA \cap \pi$ is a projective transformation (you use this map when you watch movies in cinema).

(a) Consider a 4-dimensional space V, two 3-dimensional subspaces U and U', whose projectivisations $\mathbb{P}(U), \mathbb{P}(U') \subset \mathbb{P}(V)$ are the planes π and π', and let $P = \mathbb{P}(W)$ for some 1-dimensional subspace $W \subset V$.

Show that

$$V = U \oplus W,$$

whenever $P \notin \mathbb{P}(U)$. The space is a direct sum of its two subspaces whenever each vector $v \in V$ can be *uniquely* presented as a sum $v = u + w$ for vectors $u \in U$ and $w \in W$.

Solution:

(b) Consider a linear transformation $\mathrm{proj}_U \colon V \to U$ given by

$$v = (u + w) \mapsto u.$$

Check that it is a linear transformation and its restriction to U' gives a map

$$L = \mathrm{proj}_U |_{U'} \colon U' \to U$$

of full rank under our assumptions.

Solution:

(c) Show that the prospective projection is given by $F_L \colon \mathbb{P}(U') \to \mathbb{P}(U)$ for L constructed in (b).

Solution:

(d) The proof above clearly works in any dimension. Nevertheless, prove that perspective projection from a line to a line in a projective plane is a projective transformation(i.e., fractional linear transform) only using double ratios.

Solution:

Problem 5. Show that there exists a projective transformation of $\mathbb{P}^2$ taking an arbitrary conic to a circle.

Hint: use problem 4.

Solution:

Problem 6. The goal here is to show that the projective transformation of a plane is uniquely determined by the image of 4 points (compare with 1(c), (d)).

(a) Let $X_1, \dots, X_{n+2}$ be in the general position in $\mathbb{P}(V)$, i.e. each tuple of $(n+1)$ points among them can be represented by $(n+1)$ linearly independent vectors (this is equivalent to saying that they do not belong to a projective hyperplane).

Show that there is a *unique* choice of representative vectors $v_1, \dots, v_{n+2}$ such that

$$v_{n+2} = \sum_{i=1}^{n+1} v_i.$$

Solution:

(b) Consider another tuple of points $Y_1, \dots, Y_{n+2}$ in a general position. Show that there are vectors $w_1, \dots, w_{n+2}$ representing these points and a linear transformation $T\colon V \to V$ such that

$$T(v_i) = w_i.$$

Solution:

Problem 7: applications of projective transformations and double ratios. (a) Consider a quadrangle $ABCD$ and introduce the notation $P = AB \cap CD$, $Q = BC \cap AD$, $R = AC \cap BD$, $S = PR \cap AD$. Show that $(A, B; S, Q) = -1$.

Hint: of course, double ratios on lines are preserved under projective transformations of the plane.

Solution:

(b) In the setup as above, consider a line ℓ passing through R which intersects lines AB, BC, CD, DA at points $M, X, N, Y.$, respectively Show that R is a midpoint of MN if and only if it is a midpoint of XY.

Solution:

(c) In the setup of (a), let T be the projection of R onto line PQ. Show that the opposite sides of the quadrangle are seen from T at equal angles.

Solution:

(d) **Butterfly theorem.** Chords AC and BD of a circle ω pass through the middle O of a chord MN. Chords AD and BC intersect MN at points X and Y. Show that O is the midpoint of XY.

Solution:

(e) The interior and exterior angle bisectors of $\angle A$ in a triangle ΔABC intersect BC at points L and N, respectively. Let line LM intersect AB at point P, where M is a midpoint of AC. Show that $PA = PN$.

Solution: