Financial Mathematics

ORMC Intermediate

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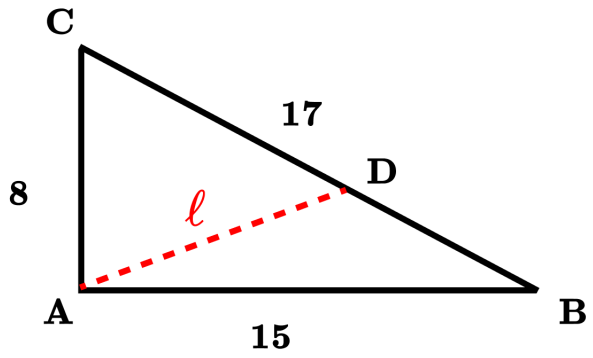
Warm-up.

(Adapted from AMC 8, 2017) Line ℓ is drawn on the right triangle $\triangle ABC$ below such that $\triangle ABD$ and $\triangle ACD$ have the same perimeter. What is the area of $\triangle ABD$?



Notice: for this packet, please use a calculator! I suggest using the website Desmos as your calculator if you can!

What's so Interesting about *Interest*?

When you start making some money from your first job, it's best not to immediately spend all that money (that's how you go broke!). But then where do you put it? Let's say we've got $1,000 to our name and consider our options.

Option 1: In your piggy bank.

You get a big piggy bank and store your $1,000 inside. After 2 years, you will have $_____.

Problem 1. Graph your total money as a function of time.

Option 2: In your dad's pocket.

You give your dad \$1,000 and he promises you he will pay you 10% of your *principal* (the initial \$1,000 you gave him) every year. After 2 years, he will give you back your principal, and you will have \$_____.

Problem 2. Graph your money as a function of time. Why do you think this is called *linear* (or more commonly, *simple*) interest?

Option 3: In a bank account.

You deposit \$1,000 into a bank account. At the end of each year, if you have $\$X$ left in your account, they will add $\$\langle 10\% \text{ of } X\rangle$ into your account. We call this 10% *compound interest*, because this means your interest also earns interest the next year. At the end of 2 years you will have \$_____.

Problem 3. Graph your money as a function of time.

As we can see, *compound interest*, the type of interest where our interest earns interest, is a powerful tool for growing our money! **Since this is the most common type of interest, when we say "interest", just assume we're talking about compound interest.**

Compound Interest.

A quick note on *percentages*.

Recall that 10% is the same as 10 per *cent*, where *cent* means "hundred". That is, $10\% = \frac{10}{100} = 0.10$ in fraction and decimal form respectively.

Problem 4. (Multiple choice) Say we want to calculate 17% of 487. Which expression shows how we can do this? (Select all that apply)

(A) 17×487

(B) 0.17×487

(C) $\frac{17}{100} \times 487$

(D) $17 \times \frac{1}{100} \times 487$

(E) $17 \times 487\%$

Computing with interest.

Note that the period of time in which interest is earned can be any length of time. Some banks pay interest every month. In general, though, if the time period is not specified, we will assume interest is paid at the end of each year.

Problem 5.

(i) Complete the table assuming an annual interest rate of $i = 10\%$.
(Note: *annual* means per year, i.e. interest gets compounded every year.)

Year	Money
0	\$1,000
1	\$1,100
2	\$1,210
3	
4	
5	

(ii) Graph the money in your account over time.

Problem 6. Now suppose you are given a table. Determine the annual interest rate.

Year	Money
0	\$20,000
1	\$20,400
2	\$20,808
3	\$21,224.16

The financial math toolkit.

In this section, we will go over the necessary math skills (our tools) needed to solve nearly all basic finance-related problems. These are *exponents, logarithms, and the geometric series sum formula!*

Accumulation function and exponents.

Let's say you deposit X dollars into a bank account with an annual interest rate of i. If t represents the number of years after year 0, we can express the amount of money in your bank account at time t:

$$A(t) = X \times (1+i)^t.$$

Note that the little ^{t} is called an *exponent*. It represents self-multiplication repeated t times. For example, $2^4 = 2 \times 2 \times 2 \times 2$. We usually pronounce 2^4 as "2 to the 4th power."

Also, when we say $1+i$, we assume that i is in decimal form. That is, if $i = 10\%$, we have that $1 + i = 1 + 0.10 = 1.10$. Let's do some practice.

Problem 7. Use a calculator for (iii) and (iv)!

(i) Compute 5^2.

(ii) Compute 7^4.

(iii) Compute 1.1^5.

(iv) If you deposit $X = \$200$ into a bank account with interest rate $i = 25\%$, how much money will your account have at time $t = 20$?

Logarithms and roots.

Here are some more tools that can help you solve equations here and in your future classes.

Problem 8 (Roots). Find x in the following equations. Hint: you can *guess-and-check* these ☺.

(i) $x^2 = 9$

(ii) $x^4 = 625$

(iii) $x^3 = 343$

(iv) $x^{11} = 2048$.

Notice that when we try to solve for an equation $x^n = k$, we are essentially "undoing" the exponent over x. We represent this mathematically by applying the symbol $\sqrt[n]{-}$ over both sides of the equation. We call this symbol the "nth root." That is, $\sqrt[n]{a}$ represents the value, which, if taken to the nth power, returns the value a. Also note that $\sqrt[2]{-}$ is always written as $\sqrt{-}$. Here's an example of us using this new notation,

$$\begin{aligned} x^5 &= 7,776 && \text{...is the same as} \\ \sqrt[5]{x^5} &= \sqrt[5]{7,776} && \text{...the left side becomes} \\ x &= \sqrt[5]{7,776} && \text{...finish with a calculator!} \\ &= 6. \end{aligned}$$

A quick note on inputing roots into calculators...

Some calculators don't have a $[\sqrt[n]{-}]$ button. But there's a workaround. Let's say you're trying to find $\sqrt[3]{27}$. It turns out that we can also write this as $27^{(1\div 3)}$

or $27^{\frac{1}{3}}$! But this can be tricky too! Ask for help if you need help with your specific calculator, and if nothing works, please use the website Desmos for an easy-to-use graphing calculator.

Problem 9 (More Roots). Practice using your calculator to find the following values. Round to the nearest hundredth.

(i) $\sqrt{100}$

(ii) $\sqrt[5]{834,257}$

(iii) $2^{0.1}$.

Problem 10. Suppose you put \$67 into your bank account and it takes 8 years to double into \$134. What is the annual interest rate of this bank account? Put your answer in percentage form rounded to 2 decimal places.

Now what if we wanted to solve an equation that looked like $2^t = 50$? We have a new function for this called the *logarithm*. I'll give an example and then explain how it works:

$$\begin{aligned} 2^t &= 50 \\ t &= \log_2(50) \\ &= 5.64. \end{aligned}$$

In general, $t = \log_a(b)$ is the value needed to make $a^t = b$. This function input can be found on Desmos.

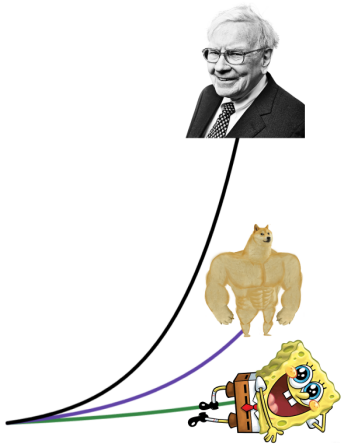
You may also find this fact useful when inputting logarithms into calculators:

$$\log_a(b) = \frac{\log(b)}{\log(a)},$$

where $\log(x)$ written without a base/subscript is also often written as $\ln(x)$. This just means it has base e, which is a number approximately equal to 2.718. Don't worry about what it represents for now, this is just to help you input logarithms into your calculator.

Now we have the tools we need to solve lots of problems! Let's go give your calculator a workout!

Problem 11 (the *power* of exponential growth).



(i) Alice puts \$80 into her savings account which has an annual interest rate of 2%. How long does it take for her money to triple?

(ii) Alice's younger sister Bernice puts \$40 into a broad market index fund which returns an annual rate of 8%. By the time Alice has tripled her money, how much money does Bernice have?

(iii) Legendary investor Warren Buffett's fund Berkshire Hathaway has famously returned 19.9% annually in the past 60 years. If you had bought \$80 of Berkshire Hathaway in 1965, how much money would you have now?

Geometric series sum formula.

Problem 12. If I place 1 grain of rice on the first box of a chess board, then 2 on the next box, then 4, 8, 16, ... and so on, will I have more than 1 trillion grains of rice at the end?

You may notice in the previous problem that you would end up with

$$1 + 2 + 2^2 + 2^3 + 2^4 + \cdots + 2^{63}$$

grains of rice. But adding all those numbers one-by-one, even with a calculator, seems annoying, to say the least. Luckily, there's a formula for adding up sums like this:

$$\frac{a(1 - r^{n+1})}{1 - r} = a + ar + ar^2 + \cdots + ar^n,$$

where a can be any number, r can be any number other than 1, and n is some positive whole number. Note that $n+1$ is the *number of terms* in the sum, and r is called the *common ratio.*

Problem 13. Find the following sums using the formula.

(i) $3 + 6 + 12 + 24 + 48 + 96$

(ii) $0.5 + 0.5(1.1) + 0.5(1.1)^2 + \cdots + 0.5(1.1)^{68}$

(iii) $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots + \frac{1}{2048}$.

Time Value of Money.

Let me tell you possibly the most important concept in finance: *The value of money is relative to time.* Think of it this way: would you want me to give you $10 now or $10 in a year? Of course you would want it now! But why?

Problem 14. Circle the following reasons for why you might want $10 now vs. in one year. There are no right or wrong answers ☺.

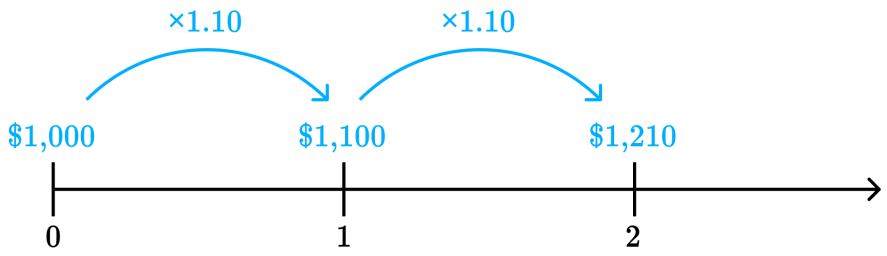
(A) I want to spend the $10 now for some ice cream!

(B) I can invest the $10 in the bank and earn interest.

(C) What if you lose/forget to give me the money in one year?

(D) Money now means more options on how to spend it than money later.

Present Value vs. Future Value.



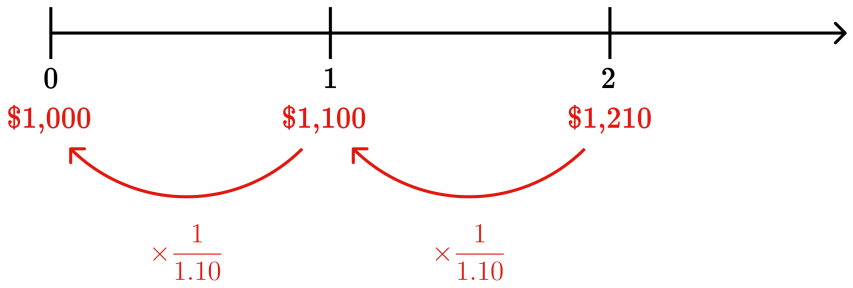
The *time diagram* is an important tool for visualizing the value of money over time. From the diagram above, we can see that if $i = 10\%$, then we can think of $1,000 now, $1,100 in one year, and $1,210 in two years as *equivalent*—at least mathematically speaking.

Try to convince yourself of this!

We call the $1,000 the *present value*, $1,100 the *time-1 value*, and $1,210 the *time-2 value*.

Discounting.

Actually, the same process is valid in reverse.



In general, if we know the annual interest rate is i, we can "advance time" forward by one year by multiplying our money by $(1+i)$ and we can "reverse time" back one year by multiplying by $\frac{1}{1+i}$ (do you see why this works?). We call these processes *accumulating* and *discounting* respectively.

Problem 15. Let's say you find an old bank account that you forgot about. They tell you that you have \$1,290.59 in your account and that your account is 6 years old. The annual interest rate is 11%.

(i) Draw a time diagram and figure out how much you had to deposit at time-0 for you to have this much money in your account today, at year 6.

(ii) You take a look at your bank statements and find that you actually made two deposits: $\$X$ at time-0, and \$500 at time-3. Draw another time diagram and find X.

Hint: how much is the balance at time-3 in terms of X? How much do you ***need*** *the balance to be at time-3 in order to arrive at time-6 with \$1,290.59?*

Reference Dates.

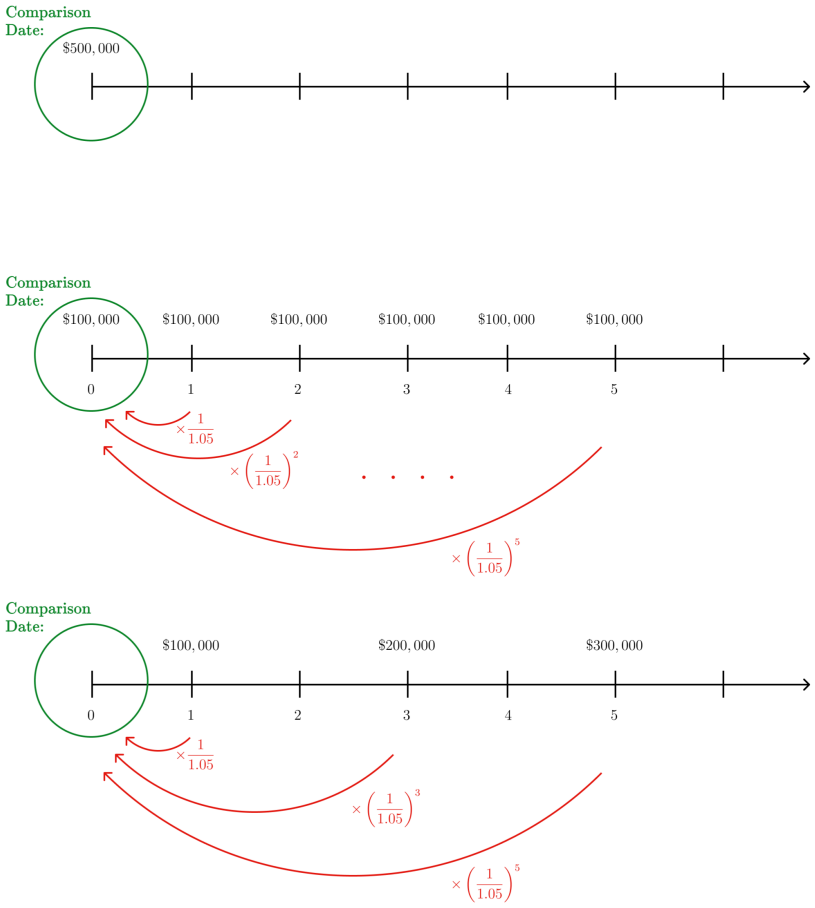
Problem 15(ii) highlights an important concept in analyzing a series of *cashflows* (think of a cashflow as any deposit or withdrawals made at a certain time).

> Before you can to add or subtract cashflows that happen at different times, you must:
>
> 1. choose a date,
> 2. accumulate all money that happens earlier than this date,
> 3. discount money that happens later than this date,
>
> so that they **all land on the same date**. We call this date the *reference date*.

EXAMPLE. George just won a jackpot! But he is now faced with choosing between three ways to claim his prize.

1. \$500,000 immediately.
2. \$100,000 for 6 years with the first payment today.
3. \$100,000 in 1 year, \$200,000 in 3 years, and \$300,000 in 5 years.

George also knows that the annual interest rate is 5%. We can figure out which choice is the best using time diagrams. Also, we choose our reference date to be time-0.



Clearly choice 1 is worth \$500,000 at time-0. But how much are choices 2 and 3 worth today? Let's calculate!

Problem 16. Continuing from our example, we note that we can discount each cashflow back to time-0 to get that

$$\begin{aligned}\text{Present Value (PV)} &= 100000 + 100000\left(\frac{1}{1.05}\right) + 100000\left(\frac{1}{1.05}\right)^2 \\ &\quad + 100000\left(\frac{1}{1.05}\right)^3 + 100000\left(\frac{1}{1.05}\right)^4 + 100000\left(\frac{1}{1.05}\right)^5.\end{aligned}$$

Calculate this value using the geometric series sum formula. Is choice 2 better than choice 1?

Problem 17. What is the present value of choice 3's cashflows?

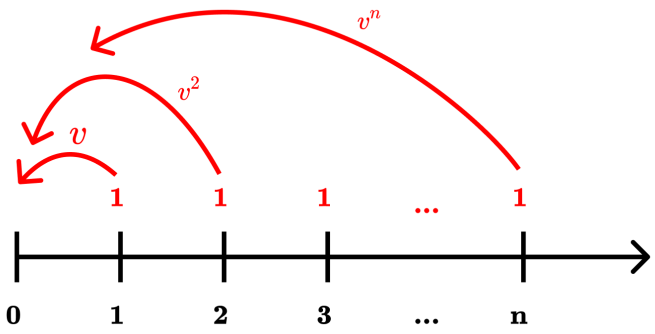
Annuities.

Annuities are simply a series of payments made regularly (e.g. every month, every year, etc.). We also define:

$$\text{Discount factor } = v := \frac{1}{1+i}.$$

The *discount factor* simply makes discounting future values easier. For example, $100v^3$ represents the value of \$100 three time periods ago.

Here are the cashflows of an annuity paying \$1 per year discounted with a discount factor of v.



Problem 18. Derive the formula

$$a_{\overline{n}|i} = \frac{1-v^n}{i}$$

for the *present value* of an annuity paying \$1 per period at an interest rate of i with the first payment in one time period. (Hint: v and i are related!).

Problem 19. Derive the formula

$$s_{\overline{n}|i} = \frac{(1+i)^n - 1}{i}$$

for the *accumulated value* (that is, the time-n value) of the same annuity.

Problem 20. If Narayanan invests \$80 every month into an S&P 500 fund returning 0.8% per month, how much money will he have in 20 years?

Problem 21. A \$1,000 face value bond will pay the bond investor 6% of the face value at the end of every year for 10 years, then pay the face value back to the investor at the end of year 10.

(i) Draw a time diagram for the cashflows of the bond.

(ii) If the interest rate during these 10 years is 4.5%, what is a fair price to purchase the bond at? (Hint: discount the cashflows back to time-0).

Homework.

Do at least two of the challenge problems below.

Challenge problems.

Some of these will require some real ingenuity, don't give up!

Problem 22. Los Angeles is introducing a new welfare system. At the end of one's first month of unemployment, the city pays $1,000. Each subsequent month, the pay is reduced by 2%. The welfare system stops after 3 years. What is the present value of this welfare benefit given that the receiver doesn't find work for 3 years, and the annual interest rate is 10%?

Problem 23. To keep employees loyal, *McAmazing-ON!* incorporated pays out yearly bonuses depending on how long an employee stays with them.

Years	Bonus
1-3	\$200
4-7	\$500
8-13	\$1,500
14-20	\$800
21+	\$10,000

If the interest rate is 8% per year, and Joe Schmoe worked at *McAmazing-ON!* incorporated for 30 years, what is the present value of all the bonuses they earned? Give the answer as an expression in terms of $a_{\overline{n}|i}$'s and v's.

Problem 24. You are a venture capitalist (VC) looking to invest into startup *Open-A-Eye*. Below are their earning expectations for the first 10 months.

Month	Projected profit
1	\$0 million
2	\$1 million
3	\$2 million
...	...
10	\$9 million

But you expect *Doodle Jammin' Eye* to enter and steal market share. The projected earnings from months 11-19 are below.

Month	Projected profit
11	\$8 million
12	\$7 million
13	\$6 million
...	...
19	\$0 million

If the monthly interest rate is at 0.5%, what is the value of this startup (how much is the maximum you are willing to pay) right now?

Problem 25. Your \$100,000 student loan is repaid with level monthly payments for 25 years at a monthly interest rate of $i = 0.93952\%$.

(i) How much is the monthly payment?

(ii) How much do you still owe after 10 years?

(iii) If after 10 years, you decide to triple your payments, how much faster do you finish paying back your loan? Round to the nearest month.