Definition 1. Define an *ellipse* as a locus of points X such that $XF_1 + XF_2 = C$ for some constant $C \geq |F_1F_2|$, where F_1 and F_2 are two fixed points, called *foci*.

Definition 2. (a) *Hyperbola* is a locus of points X such that $|XF_1 - XF_2| = C \leq |F_1F_2|$.

(b) *Parabola* is a locus of points on equal distance from a given point F (its *focus*) and a given line ℓ (its *directrix*).

Problem 1. (a) Show that the interior of the ellipse (as above) consists of points Y with $YF_1 + YF_2 < C$. In particular, a goat tied to a rope, whose ends are attached to F_1 and F_2, using a sliding ring, will eat all the grass inside the ellipse (see Figure 1).

(b) The hyperbola (as above) splits the plane into three parts. Show that points Y in the "middle" portion satisfy $|YF_1 - YF_2| < C$.

(c) State and prove a similar property for a given parabola.

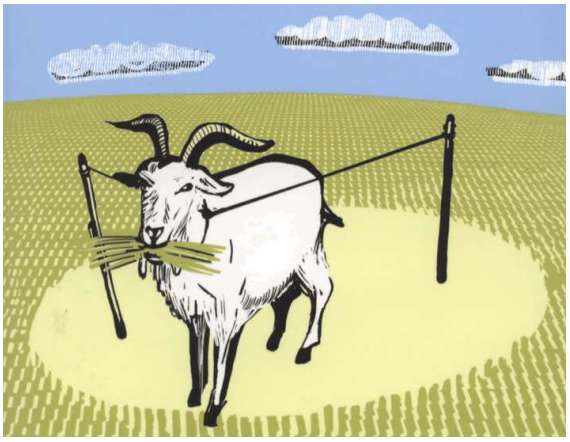

Figure 1: A goat tied to two stags.

Problem 2. (a) Given two points F_1 and F_2 on one side of a line ℓ. Find the point P with the smallest possible sum $PF_1 + PF_2$.

(b) Given two points F_1 and F_2 on different sides of a line ℓ. Find the point P with the smallest possible value of the expression $|PF_1 - PF_2|$.

(c) Given two lines ℓ and ℓ', and a point F not on these lines. Find a point P on ℓ' with the maximal value of the expression $\text{dist}(P, \ell) - PF$.

Problem 3: Optical property. (a) Let ℓ be a line tangent to an ellipse at point P, and F_1 and F_2 be the foci of the ellipse. Show that $\angle(\ell, PF_1) = \angle(\ell, PF_2)$.

(b) Formulate and prove the analogous optical properties for parabolas and hyperbolae.

Problem 4. Let PQ be a chord of an ellipse passing through its focus F_1. Let R be the intersection point of the tangent lines to the ellipse at P and Q. Show that RF_1 is perpendicular to PQ.

Problem 5: Isogonal property. (a) Let P be a point outside of an ellipse E with focal points F_1 and F_2. Let X and Y be the two points of tangency between tangent lines to E passing through P. Show that

$$\angle XPF_1 = \angle YPF_2.$$

(b) Let E be an ellipse inscribed in the triangle $\Delta A_0A_1A_2$. Show that its foci F_1 and F_2 are *isogonal conjugates*, i.e., for each vertex A_i it holds that $\angle A_{i-1}A_iF_1 = \angle A_{i+1}A_iF_2$ (the addition for indices is performed mod 3).

Problem 6. In the setup of problem 5(a), show that PF_1 is a bisector of the angle $\angle XF_1Y$.

Problem 7. Given two non-intersecting circles of radii r_1 and r_2 with centers O_1 and O_2, find the locus of the centers of circles tangent to these two circles.

Problem 8. Given a hyperbola $\mathcal{H}$ and an ellipse $\mathcal{E}$ with common foci F_1, F_2. Pick a point P on the branch of the hyperbola "closer" to F_2, outside of E. Let PF_2 and PF_1 intersect the ellipse E at points X and Y (each of them lies between P and the corresponding F_i). Show that lines XF_1 and YF_2 intersect at a point Q which also belongs to $\mathcal{H}$.

Problem 9.

(a) Given two points F_1 and F_2, find the locus of points P such that $PF_1^2 + PF_2^2 = \text{const}$. Can you solve this problem if there are n points $F_1, \ldots, F_n$?

(b) Given an ellipse E, find the locus of points P, such that the angle between the tangent lines from P to E equals $\frac{\pi}{2}$.

Problem 10. Recall that analytically, the standard presentations for ellipses, hyperbolae, and parabolas are given by

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,\ a \geq b > 0; \tag{1}$$

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1, \quad a, b > 0; \tag{2}$$

$$y^2 = 2px,\ p > 0. \tag{3}$$

For each of these presentations, show that the given equations above define ellipses, hyperbolae, and parabolas; in particular, find their foci and directrices.

Problem 11. Given an ellipse with foci F_1 and F_2, and a point P outside the ellipse. Let lines PF_1 and PF_2 intersect the ellipse at points X_1, Y_1 and X_2, Y_2, respectively. Let Q be the point of intersection of lines X_1Y_2 and X_2Y_1. Show that PQ is the bisector of the angle $\angle F_1PF_2$.

Problem 12. In the setup of Problem 8, it is also given that the perpendicular line ℓ to F_1F_2 through F_2 contains two intersection points between H and E. Show that line ℓ is the bisector of the angle $\angle XF_2Y$.