# GRAPHS IV - EULER'S THEOREM

MATH CIRCLE (INTERMEDIATE) 02/03/2013

(1) **Euler's Theorem.**
**Definition:** A graph that can be drawn in such a way that its edges do not intersect each other (except at their endpoints) is called **planar**.
**Definition:** A planar graph is **properly depicted** by a figure if its edges (as shown on the figure) do not intersect at their interior points.
If a graph is depicted properly, then it divides the plane into several regions called **faces**. Let F equal the number of faces in a graph. Let E be the number of edges, and let V be the number of vertices in that graph. The following fact is then true:

**EULER'S THEOREM: For a properly connected planar graph, the equality**

$$V - E + F = 2$$

**always holds true.**

Draw an example of a graph that is planar and one that is not planar. Now take the planar graph and depict it both properly and improperly.

(2) **Proving Euler's Theorem.**

(a) It will greatly help us to first review Problem 12, about cutting up a volleyball net, on the previous handout, Graphs III. Once the maximum number of strings has been cut, what kind of graph do the remaining ones form?

(b) When you delete an *edge* from a properly depicted graph, what happens to the number of

(i) edges? (Don't think too hard about this one.)

(ii) vertices? (Remember, we only deleted an edge, not any vertices it contained.)

(iii) faces? (It may help to draw a picture.)

(iv) The value of $V - E + F$ ?

Now that we have successfully shown that $V - E + F$ is ________________,
we are halfway done! We still need to show that this is equal to 2.

(c) Let's continue this process of deleting edges until we have deleted the maximum number such that the graph remains connected. As with the volleyball net problem, we now have a maximal tree.

(i) What is the relationship between $V$ and $E$ for this maximal tree?

(ii) What is the value of $F$ for this maximal tree?

(iii) What is the value of $V - E + F$ for this maximal tree?

(iv) How do we know that $V - E + F$ will have this same value for the original graph?

(d) Carefully re-read your solutions to parts (a)-(c). Find the assistant at your table and verbally summarize your steps for completing the above proof. You may continue onto the next problem only once your assistant approves and initials here: _________

(3) There are 7 lakes in Lakeland. They are connected by 10 canals so that one can swim through the canals from any lake to any other. How many islands are there in Lakeland?

(4) Using a picture **and** words as evidence, explain why, for a planar graph, $2E \geq 3F$.

(5) Prove that for a planar connected graph $E \leq 3V - 6$.

(6) The statement you proved in Problem 6 actually holds for *any* planar graph, whether or not it is connected. Explain why. Make sure to use the term "connected components" in your answer.

(7) The graph with 5 vertices, each of which is connected by an edge to every other, does not satisfy the inequality $E \leq 3V - 6$. However, the analogous graph with only four vertices does satisfy the inequality. Explain.

(8) Through these next few parts, our goal will be to answer this question: is it possible to build three houses and three wells, then connect each house with each well with 9 paths, no two of which intersect except at their endpoints?

(a) In the graphical representation of our situation, any cycle must alternate between house and well. Why is this?

(b) (Proof by contradiction) Let's *assume* this graph is planar and can be properly drawn. We have already shown that $2E \geq 3F$. But perhaps we can strengthen that for this situation. It turns out that, in this specific problem, $E \geq 2F$. Explain why.
(Hint: Review your solution to Problem 4, with the added stipulation introduced in part (a) of this problem.)

(c) Can you find a contradiction here? Is it possible to build the specified setup?

(9) Prove that if the degree of each of the 10 vertices of a graph is equal to 5, then the graph is not planar.

(10) Prove that in any planar graph there exists a vertex with degree no more than 5.

(11) Alice drew 7 trees on a blackboard, each having 6 vertices. Prove that some pair of them is isomorphic.

(12) Each of the edges of a complete graph with 6 vertices is colored either black or white. Prove that there are three vertices such that all the edges connecting them are of the same color.

**- Math Kangaroo Problems -**

(13) How many nonnegative whole numbers smaller than 100 can you get as a sum of nine consecutive whole numbers?

(14) In a certain month, three Tuesdays turned out to be on even days of the month. Which day of the week will be the 21st day of the month?

(15) Peter rides a bicycle from town P to town Q at a constant speed. If he increases his speed by 3 m/s, he will arrive at town Q 3 times as fast. How many times as fast will he arrive at town Q if he instead increases his speed by 6 m/s?

(16) What percent of the elements of the set of natural numbers $\{1, 2, 3, 4, ..., 10000\}$ are natural numbers?