Graph Coloring Packet

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1 Background

The next 2 weeks of class will be dedicated to answering the following question: *What is the minimum number of colors needed to be able to color every possible map, given that no bordering countries can be the same color?*

Question 1: Why would a mapmaker not want to have bordering countries be the same color?

The earliest record of this question comes from Francis Guthrie in 1852, after coloring the counties of England. (Although we originally posed this question for countries on a map, we can try and color any sort of regions. Additionally, there is no need for the map to be real.) He noticed he was able to color the counties using only 4 colors, and was unable to draw any map that required 5. However, a proof of the statement that only 4 colors were needed evaded mathematicians for over a century, until Appel and Haken of the University of Illinois were able to utilize modern computers to go through a massive amount of computation and eventually prove the theorem.

Question 2: What is a theorem? Can you think of any examples?

Although their proof was controversial at first, due to it being one of the earliest proofs to rely so heavily on computers, it would eventually become widely accepted. The math department at the University of Illinois was so proud of this proof that they would come to use the statement "Four colors suffice" as a postmark:

Due to the incredibly complex nature of their proof, we will instead prove an easier result - that every map can be colored with 6 colors. Even though no map requires 6 colors, this serves as an upper bound.

2 Coloring some maps

Before we start coloring, there a couple definitions we must clarify. Firstly, two countries are only considered to border each other if they share some nonzero length on their border - simply meeting at a point doesn't constitute bordering. Otherwise, we could have 10 countries arranged like a pie and they would all border each other because of the central point, and that would take 10 colors. With our definition, each slice only borders its two neighbors.
Secondly, every country must be *contiguous*.

Question 3: What does contiguous mean? Is the United States a contiguous country? Discuss with your instructors.

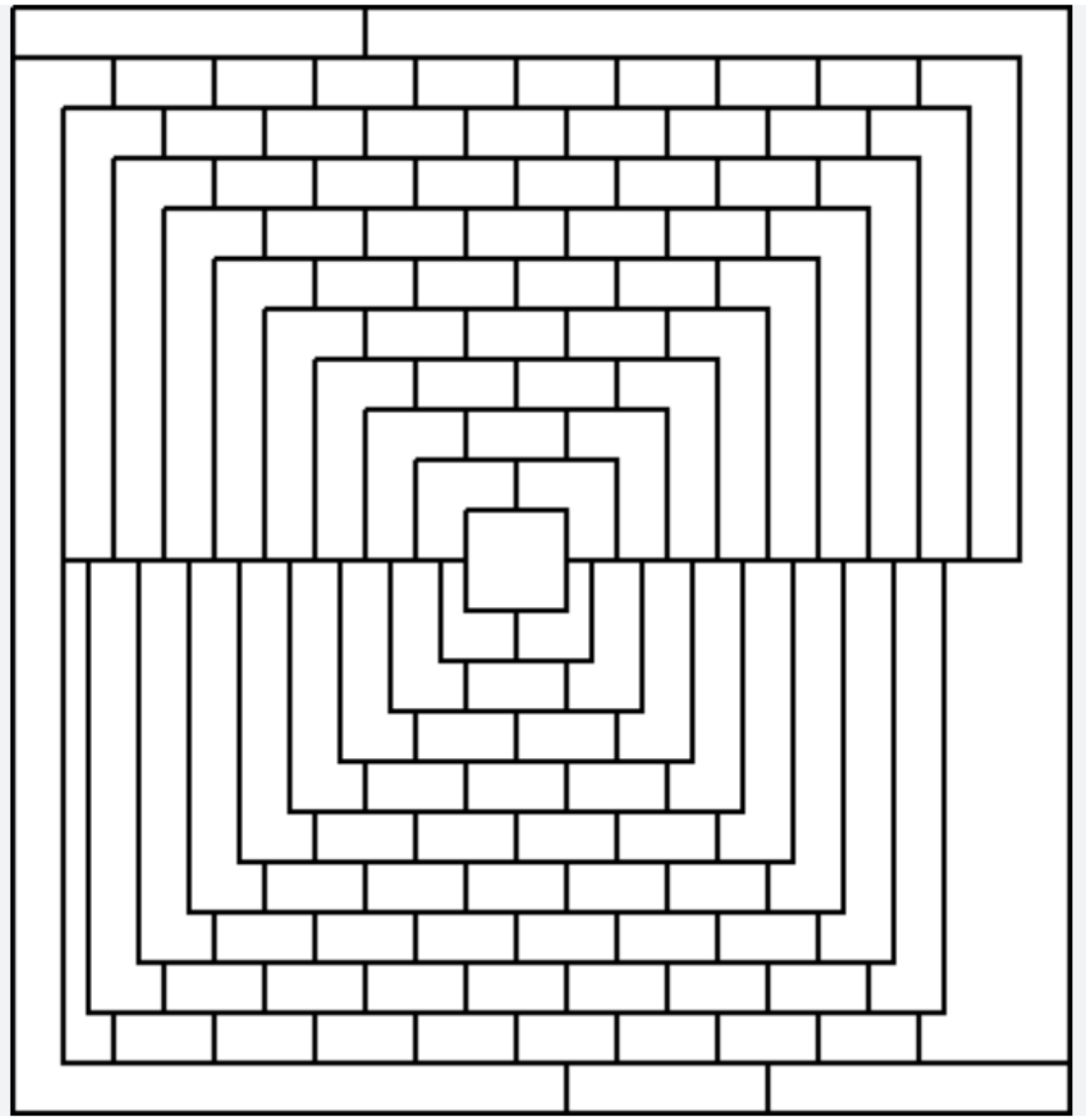
Now we will begin coloring. If you don't have any erasable colored pens, you can just use a number for each color.

Question 4: Color the mainland United States:

Question 5: Color the counties of England:

Question 6: Draw your own map that requires 4 colors. Can you do this using only 4 countries?

Question 7 (red pepper): For April Fools of 1975, Martin Gardner published the following map and claimed it required 5 colors. Can you color it with 4?



3 Graphs

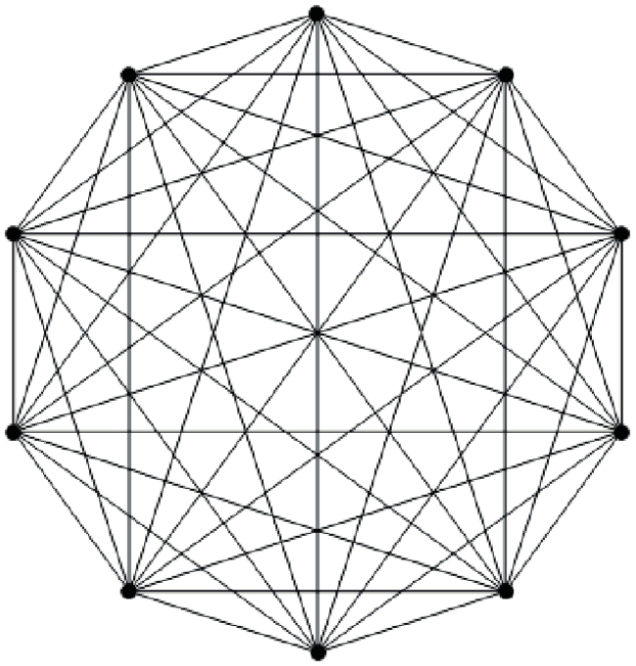
A *graph* is defined as a group of dots that we call *vertices* and lines between the dots that we call *edges*. We do not care about the location of the vertices or the length of the edges - the edges need not even be lines! All that matters for our purposes is which vertices are connected (we will also have to worry about whether or not edges cross each other later). If two vertices are joined by an edge, we call them *adjacent.* Additionally, we will never have more than one edge between two vertices, and a vertex cannot have an edge with itself (a graph with these properties allowed might be called a *hypergraph*).

Question 8: Alice, Bob, Charles, and Daniel play a tournament where everyone plays each other, except Charles does not play Daniel. Draw a graph where each person is a vertex and two people are connected with an edge if they played a game. How many games were played?

We will use graphs to help simplify our maps. Within our graphs, every country will be a vertex, and we will draw an edge between two countries if they border each other. We must assign colors to each vertex such that no two adjacent vertices are the same color.

Question 9: Draw a graph representing this map of the states of Australia, and color the graph:

It is fairly clear that we can represent any map with a graph. However, does every graph correspond to a map? Consider the following graph, 10 vertices with every possible edge:



There's no possible arrangement of 10 countries that has each country bordering every other country - there's simply not enough space! We will have to impose another restriction on the types of graphs we consider.

Definition: A graph is called *planar* if no two edges intersect each other.

Think about why planar graphs represents our maps.

Question 10: Draw 4 vertices with all possible edges such that no two edges cross each other. Conclude this is a planar graph, and draw a map corresponding to this graph. Explain why this means 3 colors would not suffice to color every possible map.

For the remainder of class, we will try and find a relationship between the number of edges, vertices, and *faces* of a planar graph with some experimentation. Faces of a graph are the number of regions the edges create. The entire unbounded "outside" region is also counted as a region.
Discuss why we must require a graph to be planar for us to be able to discuss its faces.

Question 11: Look at the graphs you drew for questions 8 and 9. Count the number of vertices, edges, and faces.
There is one additional condition we will impose on our graphs for convenience - the reason will become clear next class. All the planar graphs you draw must be *connected*, meaning between any two vertices there is some path of edges that connects them.

Question 12: Using the backsides of the paper in this packet, draw some planar graphs. Try and draw the most different and extreme examples you can think of while staying within our rules, and not getting too large and making it tedious to count faces. For your convenience, make you drawings big, clear, and clean. For each graph, count the number of vertices, edges, and faces. Document them in this table, where each row represents the counts of one graph. **Can you find any formula involving the addition/subtraction of vertices, edges, and faces that seems to be constant across all your graphs?** This will be the most important question for our discussion in our final class.

Vertices	**Edges**	**Faces**