LAMC Junior Circle **February 3, 2013**

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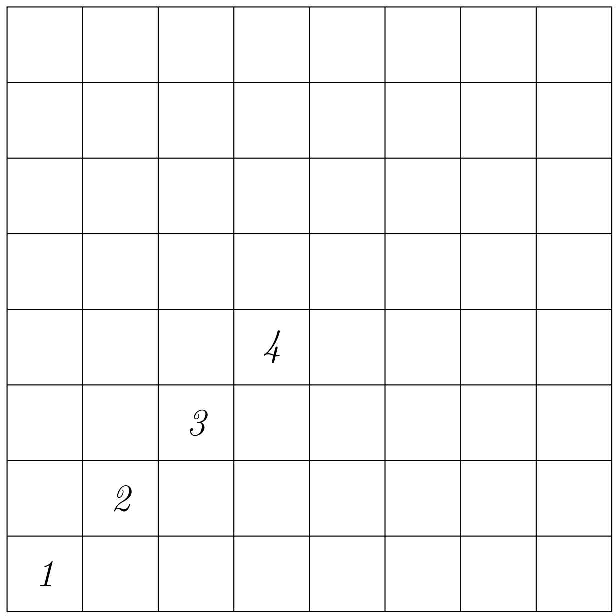
Warm-up

Problem 1 *Compute the following.*

$$-2 - 3 \times (-4) + \frac{6}{-2} =$$

Problem 2 *Can the value of a fraction increase, if we add one to the numerator and one hundred to the denominator?*

Problem 3 *Cut the board below into four equal parts so that each part contains one of the four digits.*



Problem 4 *A student has read 138 pages which is 23% of the book she is reading. How many pages are there in the book?*

Recall that the numbers $\mathbb{N} = \{1, 2, 3, \ldots\}$ are called *natural*.

A natural number greater than one is called *prime* if it is divisible only by itself and by one. For example, 3 is a prime number. A natural number that is not prime is called *composite*. For example, 6 is a composite number as $6 = 2 \times 3$.

You may find the following facts useful for the problem that follows.

$$203 = 7 \times 29,\ 209 = 11 \times 19,\ 211 \text{ is prime}$$

Problem 5 *Mary plays the following game. Starting with 1, she adds a natural number to the list if it can be obtained by increasing any of the numbers already in the list by any % from 1 through 100. For example, Mary adds 2 to the list because 2 is the 100% increase of 1. Mary adds 3 to the list as a 50% increase of 2. Mary adds 4 to the list as a 100% increase of 2. Mary adds 5 to the list as a 25% increase of 4, and so on.*

$$1 \xrightarrow{100\%} 2$$

$$2 \xrightarrow{50\%} 3$$

$$2 \xrightarrow{100\%} 4$$

$$4 \xrightarrow{25\%} 5$$

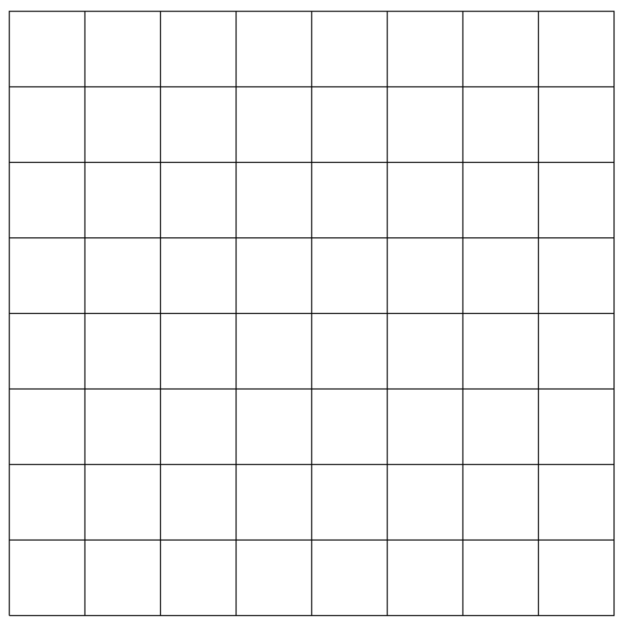
$$3 \xrightarrow{100\%} 6$$

$$4 \xrightarrow{75\%} 7 \ldots$$

What is the smallest natural number that will not appear in Mary's list?

Problem 6 *There are 11 people in a soccer team. How many ways are there to elect a captain and his second?*

Problem 7 *How many ways are there to put two rooks on a chess board so that they cannot take each other ...*



a. *... if the rooks are of different colors?*

b. *... if the rooks are of the same color?*

Some basic enumerative combinatorics

Problem 8 *How many ways are there to put three marbles of different colors in a row?*

Definition 1 *The product of all the natural numbers from 1 through n is called n factorial.*

$$n! = 1 \times 2 \times 3 \times \ldots \times n$$

For example, $3! = 6$. It is a useful convention to set $0! = 1$.

Problem 9 *Compute the following numbers.*

$a.\quad 5! =$ $\qquad$ $b.\quad 6! =$

Problem 10 *How many ways are there to put $n+1$ marbles of different colors in a row?*

Problem 11 *Compute the following numbers.*

$a.\quad \dfrac{5!}{4!} =$ $\qquad$ $b.\quad \dfrac{100!}{98!} =$

Problem 12 *There are 10 marbles of different colors in a box. How many ways are there to put 6 of them in a row? Write down the answer using the $n!$ notation.*

Problem 13 *Let k and n be natural numbers such that $k \leq n$. There are n marbles of different colors in a box. How many ways are there to put k of them in a row?*

Definition 2 *A way to choose k objects out of n so that the order of the chosen objects matters is called a permutation.*

As proven in Problem 13, the number of permutations is given by the following formula.

$$P(n,k) = \frac{n!}{(n-k)!} \tag{1}$$

Definition 3 *A way to choose k objects out of n so that the order of the chosen objects does not matter is called a combination.*

Example 1 *A poker hand is any combination of 5 cards out of the deck of 52. Let us compute the number of the possible poker hands. First, there are*

$$P(52,5) = \frac{52!}{(52-5)!} = 48 \times 49 \times 50 \times 51 = 311,875,200$$

ways to put 5 cards out of 52 in a row. However, the order of the cards in a hand does not matter. There are 5! *ways to order 5 cards. (Why?) Dividing the above number by* 5! *gives us the number of the possible poker hands.*

$$C(52,5) = \frac{P(52,5)}{5!} = \frac{52!}{(52-5)!\,5!} = 2,598,960$$

A more modern, but less convenient, notation for the number of combinations is given on the left-hand side of the following formula.

$$\binom{n}{k} = \frac{n!}{(n-k)!\,k!} = C(n,k) \tag{2}$$

The above reads as *n choose k*. For the reason that will become clear in Problem 23, the formula is also known as a *binomial coefficient*.

Problem 14 *Compute the following numbers.*

a. $\binom{5}{0} =$

b. $\binom{5}{1} =$

c. $\binom{10}{3} =$

d. $\binom{10}{7} =$

Problem 15 *Prove the following property of the binomial coefficients.*

$$\binom{n}{k} = \binom{n}{n-k} \tag{3}$$

Problem 16 *Prove the following formulae.*

a. $\binom{4}{2} + \binom{4}{3} = \binom{5}{3}$

b. $\binom{5}{3} + \binom{5}{4} = \binom{6}{4}$

Problem 17 *Prove that the following formula holds for any $n \in \mathbb{N}$ and for any $k = 0, 1, 2, \ldots, n$.*

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k} \tag{4}$$

Formula 4 explains why it is possible to arrange the binomial coefficients in the following table known as *Pascal's triangle*.

$n = 0$:						1					
$n = 1$:					1		1				
$n = 2$:				1		2		1			
$n = 3$:			1		3		3		1		
$n = 4$:		1		4		6		4		1	
$n = 5$:	1		5		10		10		5		1
$n = 6$:											
$n = 7$:											

Problem 18 *In the table above, fill in the entries of the Pascal's triangle for* $n = 6$ *and* 7.

The upper-case Greek letter Σ (Sigma) is used in math as a notation for a sum. For example,

$$\sum_{k=0}^{n} a_k x^k = a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n.$$

Problem 19 *Expand the following formula.*

$$\sum_{k=0}^{3} \binom{3}{k} x^{3-k} y^k =$$

Problem 20 *Expand and simplify the following.*

$(x+y)^3 =$

Compare your result to that in Problem 19.

Problem 21 *Expand the following formula.*

$$\sum_{k=0}^{4} \binom{4}{k} x^{4-k} y^k =$$

Problem 22 *Use the answer to Problem 20 to expand and simplify the following.*

$(x+y)^4 =$

Compare your result to that in Problem 21.

Problem 23 *Prove the following statement known as the binomial formula.*

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k}\, y^k = \\ x^n + nx^{n-1}y + \tbinom{n}{2}x^{n-2}y^2 + \tbinom{n}{3}x^{n-3}y^3 + \ldots + y^n \tag{5}$$

Note that the corresponding line of the Pascal's triangle gives you all the binomial coefficients for the binomial of power n.

Problem 24 *Use the binomial formula to expand the following polynomial.*

$$(x+1)^5 =$$

Problem 25 *Use the Pascal triangle to find the sum of all the binomial coefficients for the following n.*

$$n = 0 :$$

$$n = 1 :$$

$$n = 2 :$$

$$n = 3 :$$

$$n = 4 :$$

$$n = 5 :$$

Can you guess the general pattern?

Problem 26 *Prove the following formula.*

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n \tag{6}$$

Back to probability

Problem 27 *We toss a coin 4 times. How many ways are there to get the following outcomes?*

0 heads:

1 head:

2 heads:

3 heads:

4 heads:

Compare the above numbers to the Pascal's triangle line for $n = 4$. *Can you explain what you see?*

Suppose that the coin you toss is not necessarily fair. At every toss, the chance to get a head is p.

Problem 28 *What is the chance to get a tail?*

Problem 29 *You toss the above coin 7 times. What is the chance that you get 3 heads?*

Problem 30 *You toss the above coin 7 times. What is the chance that you get no more than 3 heads?*

Problem 31 *Assume that the coin in Problem 30 was a fair one. What is the probability that you get no more than 3 heads?*

Problem 32 *You toss a fair coin 12 times. What is the chance that you get 5 tails?*

Problem 33 *To win a jackpot in the Mega Millions lottery, you have to guess right 5 numbers from a pool of numbers from 1 to 56 and an additional Mega Ball number from a second pool of numbers from 1 to 46. What is a chance to hit the jackpot, if you purchase one lottery ticket?*

Problem 34 *The price of one lottery ticket for the Mega Millions lottery is $1. How much money do you need to invest to have a* 0.5 *chance of winning the jackpot?*