LAMC Intermediate I & II

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The Josephus problem

A Jewish general turned Roman historian, Josephus Flavius[1], lived in the first century AD. His book *The Judean War* describes a Jewish rebellion against the Roman Empire that ended up with destruction of Jerusalem in AD 70.

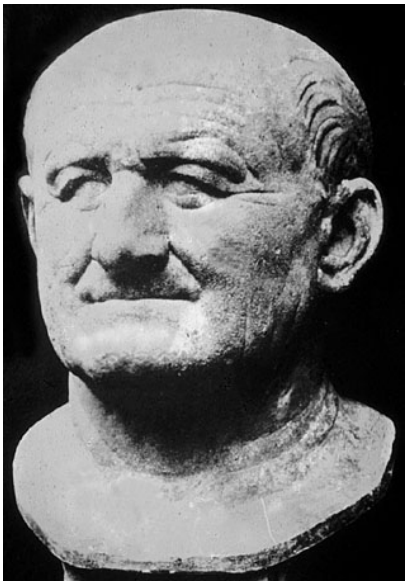
During the war, Josephus and his forty soldiers were trapped in a cave, with Romans blocking the exit. The Jews chose suicide over capture. Since suicide is not allowed in Judaism, the Jews decided to proceed the following way. They formed a circle. Then a fighter killed his neighbor on the right with a blow of his sword and passed the sword further right. They kept doing so until there was one man left standing. That one had no other choice but to kill himself. The last one happened to be Josephus. Instead of committing suicide, he turned himself over to the Romans.[2]

Josephus proceeded to become a friend of the Roman emperor Vespasian Flavius (hence Josephus's Latin last name) and a popular historian.

[1] born Yosef ben Matityahu

[2] This version of the story is modified. You will find the actual version at the end of the packet.

Josephus Flavius



Vespasian Flavius

The above story gave rise to a family of problems all bearing the common name, the *Josephus Problem*.

Josephus and his forty soldiers were trapped in a cave. This means that there was a total of 41 fighters in the circle. Let us number 1 the first fighter to raise his sword, let us number 2 the fighter to his right, etc. The goal of this lesson is to solve the following two problems.

Problem 1 *What was the position of Josephus in the circle?*

Problem 2 *Suppose that there are n soldiers, including Josephus, in the cave. What should the position of Josephus be in order for him to stay alive?*

Let us call $J(n)$ a solution of problem 2. Then to solve problem 1, we need to find $J(41)$. The solution of problem 2 is split into steps in the lesson below.

Example 1 *It is very simple to compute $J(2)$. The first fighter kills the second. Since there were only two of them, $J(2) = 1$. We show that the second fighter was killed by crossing out the number 2 in the corresponding row of the table below.*

If there are three fighters in the cave, the first kills the second and gives the sword to the third. The third proceeds to kill the first. Thus, $J(3) = 3$. To show the final stage of the process, the numbers 1 and 2 are crossed out in the corresponding row of the table below.

n	$J(n)$	
2	*1*	*1* ~~*2*~~
3	*3*	~~*1*~~ ~~*2*~~ *3*

Problem 3 *Show that $J(n)$ cannot be an even number.*

Problem 4 *Show that if $n = 2^m,\ m \in \mathbb{N}$, then $J(n) = 1$.*

Problem 5 *On a separate paper sheet, continue the above table for $n = 4, 5, \ldots, 20$. Observe the pattern and try to guess the $J(n)$ formula for any $n \in \mathbb{N}$.*

Problem 6 *Use induction to prove that the pattern you have discovered works for any $n \in \mathbb{N}$.*

Problem 7 *For $n = 2, 3, \ldots, 10$ write n next to $J(n)$ using binary numbers. Find the pattern and formulate the algorithm for getting $J(n)$ out of n in the binary notations.*

Problem 8 *Find $J(41)$.*

Generalization

Suppose that the first fighter kills not his immediate neighbor, but the k-th person to the right and passes the sword to the $k+1$-st person. Let us call $J(k, n)$ the set of numbers of the fighters that survive the group suicide. Note that the number $J(n)$ figured out in this lesson is the only element of the set $J(1, n)$.

Problem 9 *How many elements are there in the set $J(k, n)$?*

In the original story from *The Judean War* book, the set was $J(2, 41)$, not $J(1, 41)$. Josephus left the cave not alone, but with a friend.

Homework

Problem 10 *Use a programming language of your choice to write a program that computes $J(k, n)$ for any n and $k < n$.*

Problem 11 *Find $J(2, 41)$.*