Algebra and Induction

ORMC - Advanced - Summer 2025

Due July 13, 2025

1 Algebra

Consider the following expressions. All can be solved without brute force computation. And remember to show your work!

Problem 1. Suppose $x + \frac{1}{x} = 3$. Compute the value of

$$x^2 + \frac{1}{x^2}$$

Problem 2. Simplify the following:

(a)

$$\frac{x^2 - 4x}{x^2 - 2x - 8}$$

(b)

$$(x+5)^2 - (x-5)^2$$

2 Induction

Problem 3. Say you're proving that some statement $P(n)$ is true for all $n \geq 1$ using induction. If you have shown that the **inductive step** is valid but have not verified the **base case**, why is this problematic?

Problem 4. What is the smallest value of n that can be used as a base case in a proof by induction to show that $2^n > n^2$? *Hint: You do not need to prove the full result (but feel free to do so!). Just determine the smallest valid value of n.*

Problem 5. Using induction, prove the following. Make sure to explicitly show the base case and inductive step!

(a)

$$1^2 + 2^2 + 3^2 + ... + n^2 = \frac{n(n+1)(2n+1)}{6}$$

(b)

$$2 + 2^2 + 2^3 + ... + 2^n = 2(2^n - 1)$$