Mathematical Induction, Peano Axioms, ORMC Meeting Adv

06.29.25 Meeting

Author: Francis Park

1 Introduction

Hey! If you are reading this I hope you had a great first session today; I definitely did [1]. I'll be making these packets and releasing them a day or two after each session. This is mainly so that you all can go over problems you could not figure out during our time together or if you wanted to compare your solutions to mine. If you think a solution of mine is wrong, let me know – you can make fun of me during our next meeting or send me an email but put ORMC in the tag/title line of something. [2] Also if you would like these solutions are explanations to be more detailed, or you think they are too explicit, or you just want to leave me with suggestions on how I can help you learn better, that sort of feedback would be very much appreciated :)

Ideally you ask me or any of the other instructors for help during class, but I get that everybody learns differently and sometimes 2 hours is not enough to get through all the problems. Before looking at my solutions, **I really encourage you to try out the problem once more** and feel free to look at the hints (okay, not all of them have hints :/) I provide before going over my solution to each problem. That's enough rambling from me.

2 Induction

2.1 §2.1

Recall that we said that the sequence $p_1 = a$, $p_2 = a+q$, .. $p_n = a+(n-1)q$ is called an *arithmetic sequence*. Problem **2.1** asks you to derive

$$\sum_{i=1}^{n} p_i = a + (a+q) + \dots (a+(n-1)q) = na + q\frac{n(n-1)}{2} \tag{1}$$

from the following formula that we proved in class:

$$1 + 2 + \cdots + n = \frac{n(n+1)}{2} \tag{2}$$

2.1.1 Hint(s) for 2.1

1. Can you think of a way to cleverly rewrite (1) so that we can apply (2)? You want to try and break down (1) into its *simplest constituents*, if that makes any sense.
2. Here's a bigger hint: *Try and count the number of a's and q's that are being summed*

[1] This is me from the past, but still

[2] francispark@g.ucla.edu

2.1.2 Solution 1 to 2.1

Proof. First, expand the summation and write out the expression in full: $a+(a+q)+\ldots a+(n-1)q$. Notice that we are summing a n-times and then we are summing $q+2q+\ldots(n-1)q$, which is $q(1+2+\cdots+n-1)=q\frac{n(n-1)}{2}$ given that we get by applying (2) directly. □

2.2 §2.2

Use mathematical induction to prove (1)

Proof. Note that arguments like these have a rigid structure; we must first establish the base case. **Base case:** Note that $\sum_{i=1}^{1} p_i = a = (1)a + q\frac{(1)(1-1)}{2}$ and this verifies the base case, which you may liken to knocking over the first domino. **Inductive hypothesis:** Let us assume that $\sum_{i=1}^{n} p_i = na + q\frac{n(n-1)}{2}$ holds for some n. We wish to show that the latter assumption implies that the analogous statement holds for $p(n+1)$. We may proceed as follows:

$$\sum_{n=1}^{n+1} p_i = \sum_{i=1}^{n} p_i + p_{n+1} = na + q\frac{n(n-1)}{2} + a + nq \tag{3}$$

$$= (n+1)a + q(\frac{n(n-1)}{2} + \frac{2n}{n}) \tag{4}$$

$$= (n+1)a + q\frac{n(n+1)}{2} \tag{5}$$

and this computation concludes the argument (for we have shown $p(n) \implies p(n+1)$). □

2.3 §2.3

Use (1) for the sum of an arithmetic sequence to prove the following identity:

$$1+3+5+\ldots(2n-1) = n^2 \tag{6}$$

2.4 Hint(s) for §2.3

Identify what a and q are in this case

2.4.1 Solution to §2.3

Proof. Setting $a=1, q=2$, we get that

$$1+3+5+\ldots(2n-1) = n + 2\frac{n(n-1)}{2} = n + n^2 - n = n^2 \tag{7}$$

□

2.5 §2.4

Consider the pictures below (see the packet) below to prove the previous formula using geometry rather than algebra.

2.5.1 Hint(s) for §2.4

Fix your attention, at least for now, to the 3×3 cube. The third term in our sum corresponds to the 5 shaded sub-cubes; we have four sub-cubes remaining. What are other terms are we summing before 5? Can you see a pattern? If you do, can you generalize?

More explicitly, the first three terms in the sum are $1, 3, 5$. Do the shaded regions of the 1×1 cube and 2×2 cube be used to *shade in* the un-shaded region in the 3×3 cube?

2.5.2 Solution for §2.4

proof sketch: Consider the sum $1 + 3 + 5 + \cdots + (2n-1)$. Notice that $2n - 1 = n^2 - (2n-1)$, which, geometrically, can be viewed as the unshaded region of an $n \times n$ cube. This unshaded region (of the $n \times n$ cube) has area $(n-1)^2$. The remaining terms in the sum can be used to fill in this region, which means the total sum is simply n^2, as desired. To be super rigorous, you could argue inductively.

2.6 §2.5

Prove the above formula

$$1 + 3 + 5 + \cdots + (2n-1) = n^2 \tag{8}$$

using induction.

Proof. It is easy to establish the validity of the base case $n = 1$ (if you are unsure, explicitly compute both sides and check for equality); let us posit that the above formula holds for some n. Our goal now is to deduce that

$$1 + 3 + 5 + \cdots + (2n-1) + (2n+1) \underbrace{=}_{?} (n+1)^2 \tag{9}$$

Note that

$$\underbrace{1 + 3 + 5 + \cdots + (2n-1)}_{=n^2} + (2n+1) = n^2 + 2n + 1 = (n+1)^2 \tag{10}$$

and this finishes the argument. □

2.7 §2.6

Use mathematical induction to prove the following:

$$\sum_{i=1}^{n} i^2 = 1^2 + 2^2 + 3^2 + \ldots n^2 = \frac{n(n+1)(2n+1)}{6} \tag{11}$$

Proof. It is easy to check that the base case holds (the first domino falls). Thus, let us inductively assume that the above sum holds for some n. The goal now is to establish the validity of the following:

$$\sum_{i=1}^{n+1} i^2 \underbrace{=}_{?} \frac{(n+1)(n+2)(2n+3)}{6} \tag{12}$$

The following series of computations yields the desired conclusion:

$$\sum_{i=1}^{n+1} i^2 = \sum_{i=1}^{n} i^2 + (n+1)^2 = \frac{n(n+1)(2n+1)}{6} + \frac{6(n+1)^2}{6} \tag{13}$$

$$= \frac{(n+1)\Big(n(2n+1)+6(n+1)\Big)}{6} = \frac{(n+1)(2n^2+7n+6)}{6} = \frac{(n+1)(2n+3)(n+2)}{6} \tag{14}$$

and we are done. □

2.8 §2.7

Experiment with sums of cubes of the form $1^3+2^3+3^3+\ldots n^3$ for various natural numbers n. Try to find a formula similar to the ones we derived previously. Then, use mathematical induction to prove it.

I will provide you with two solutions; one which I came up with myself (and requires induction) and another that I encountered in my group theory class taught by Professor Greene last fall.

2.8.1 Solution 1 to §2.7

My approach is always to tinker a little bit, play around with the sums, and find a pattern. Let's start off small, but let's skip looking at 1^3 because I mean that's not too interesting.

$$1^3+2^3 = \boxed{9};\quad 1^3+2^3+3^3 = \boxed{36};\quad 1^3+2^3+3^3+4^3 = \boxed{100}; \tag{15}$$

You see the pattern right? We are progressing through certain square numbers. We formally [3] deduce that

$$\boxed{1^3+2^3+3^3+\ldots n^3 \underbrace{=}_{?} (1+2+3+\cdots+n)^2 = (\frac{n(n+1)}{2})^2 = \frac{n^2(n+1)^2}{4}} \tag{16}$$

The proof via induction is left as an exercise for the reader.

2.8.2 Neat solution method to §2.7

We sketch an approach (suggested by Professor Greene) that will allow the reader to derive closed form formulas to $\sum_{k=1}^{n} k^j$ for any $j \in \mathbb{Z}^+$ in an inductive manner.

2.8.3 Deriving $\sum_{i=1}^{n} i^2$

Look at

$$(N+1)^3 - N^3 = 3N^2 + 3N + 1 \tag{17}$$

and sum the LHS (left hand side) from $N=1$ to $N=n$. Notice that we have at our disposal $\sum_1^n 1 = \frac{n(n+1)}{2}$. Like always, however, let's start out small and experiment a little. Take $N=3$. Then, we have the following equations:

$$(1+1)^3 - 1^3 = 3(1)^2 + 3(1) + 1 \tag{18}$$

[3] We have not proved this yet! This is what we will prove using induction

$$(2+1)^3 - 2^3 = 3(2)^2 + 3(2) + 1 \tag{19}$$

$$(3+1)^3 - 3^3 = 3(3)^2 + 3(3) + 1 \tag{20}$$

The key is to notice to add the equations as desirable cancellation occurs [4].

$$(3+1)^3 - 1^3 = 3(\sum_{N=1}^{3} N^2) + 3(\sum_{N=1}^{3} N) + \sum_{N=1}^{3} 1 \tag{21}$$

Generalizing this process for N gives us the following when we let $N = n$:

$$(n+1)^3 - 1^3 = 3(\sum_{N=1}^{n} N^2) + 3(\sum_{N=1}^{n} N) + \sum_{N=1}^{n} 1 \tag{22}$$

Solving for $\sum_{N=1}^{n} N^2$,

$$\sum_{N=1}^{n} 3N^2 = (n+1)^3 - 1^3 - 3\sum_{N=1}^{n} N - \sum_{N=1}^{n} 1 \tag{23}$$

Solving gives us that

$$3\sum_{N=1}^{n} N^2 = (n+1)^3 - 1 - 3(\frac{n(n+1)}{2}) - n \tag{24}$$

$$6\sum_{N=1}^{n} N^2 = 2(n+1)^3 - 3n^2 - 5n - 2 = n(n+1)(2n+1) \tag{25}$$

$$\sum_{N=1}^{n} N^2 = \frac{n(n+1)(2n+1)}{6} \tag{26}$$

2.8.4 The generalization

This is what Professor Greene asked us: *Can you do the process in the above problem for higher powers (inductively on the power)? How does it work for 3rd powers, i.e.* $\sum_1^n N^3 =$?

Proof. Yes, this process can be done inductively. To determine $\sum_{n=1}^{n} N^3$ we proceed as follows: First, it is fruitful to examine the following quantity:

$$(N+1)^4 = N^4 + 4N^3 + 6N^2 + 4N + 1 \tag{27}$$

which can be computed directly or quickly via the binomial theorem (and knowledge of pascal's triangle). Then, notice that the following development is similar to the method in the above problem:

$$(N+1)^4 - N^4 = N^3 + 6N^2 + 4N + 1 \tag{28}$$

Summing both sides from $N = 1$ to $N = n$ gives us that

$$(n+1)^4 - 1^4 = 4\sum_{N=1}^{n} N^3 + 6\sum_{N=1}^{n} N^2 + 4\sum_{N=1}^{n} N + \sum_{N=1}^{n} 1 \tag{29}$$

[4] If you want to sound cool this is referred to as *telescoping*

and we can solve for $\sum_1^n N^3$ as follows:

$$4\sum_{N=1}^{n} N^3 = (n+1)^4 - 1 - 6(\frac{n(n+1)(2n+1)}{6}) - 4(\frac{n(n+1)}{2}) - n \tag{30}$$

$$4\sum_{N=1}^{n} N^3 = n^4 + 2n^3 + n^2 = n^2(n+1)^2 \tag{31}$$

$$\sum_{N=1}^{n} N^3 = \frac{n^2(n+1)^2}{4} \tag{32}$$

□

2.9 Remarks

This, in my opinion, is an elegant solution because it allows us to push our results further and generalize. It is powerful. Yes, we had to do more work (algebra) but when we want to examine higher power sums, you will notice quickly that spotting patterns and tricks becomes unfeasible. I want to encourage you to, even after you have solved a problem, think back and ask yourself [5]: *Can I generalize? Is there a more elegant solution?* Also, just going back and playing around with a problem you have already solved is actually quite rewarding as well.

2.10 §2.8

The following remarkable formula holds for the sum of the fourth powers of the first n natural numbers.

$$\sum_{i=1}^{4} i^4 = 1^4 + 2^4 + \cdots + n^4 = \frac{n^5}{5} + \frac{n^4}{2} + \frac{n^3}{3} - \frac{n}{30} \tag{33}$$

Prove (33).

Do you see why we introduced a generalized procedure above? It's for situations like these. That being said, a proof by induction to the problem is *completely valid* and if you have time you should definitely work through the algebra (you have to expand a bunch of terms). What I think would be more fun would be to try and use the method discussed above to derive the nasty expression you see above.

References

[5] Something I have to work on as well!