ORMC Advanced 1 **May 18th, 2025**

Oleg Gleizer
prof1140g@math.ucla.edu

Warm-up

Recall that cryptarithmetic, also know as cryptarithm, alphametics, and word addition, is a math game of figuring out unknown numbers represented by words. Different letters correspond to different digits. Same letters correspond to same digits. The first digit of a number cannot be zero.

Problem 1 *Solve the following cryptarithm, in German.*

$$\begin{array}{r} E\ I\ N\ S \\ E\ I\ N\ S \\ +\ E\ I\ N\ S \\ E\ I\ N\ S \\ \hline V\ I\ E\ R \end{array}$$

Recall that two figures in the (Euclidean) plane are called *similar*, if they have the same shape, but possibly different size.

Problem 2 *Draw a pair of similar hexagons in the space below.*

Problem 3 *One day, Oleg drew two similar hexagons on a paper sheet and cut them out with scissors. Oleg was quite surprised to find out that the larger hexagon never completely covered the smaller one no matter how he moved the figures on the table. Draw a pair of similar hexagons that have this property.*

Back to Boolean algebra

Problem 4 *Prove that in Boolean algebra addition is distributive with respect to multiplication.*

$$A + (B \times C) = (A + B) \times (A + C)$$

A	B	C	$B \times C$	$A + (B \times C)$	$A + B$	$A + C$	$(A + B) \times (A + C)$
0	0	0					
1	0	0					
0	1	0					
0	0	1					
1	1	0					
1	0	1					
0	1	1					
1	1	1					

Negation of composite statements

The two formulae proven in Problems 5 and 7 below are fundamental for understanding the algebra of logic.

Problem 5 *Prove that* $\neg(A+B) = \neg A \times \neg B.$

A	B	$A+B$	$\neg(A+B)$	$\neg A$	$\neg B$	$\neg A \times \neg B$
0	0					
1	0					
0	1					
1	1					

Problem 6 *Negate the following statement.* **My dad likes to watch football or baseball.**

Problem 7 *Prove that* $\neg(A \times B) = \neg A + \neg B$.

A	B	$A \times B$	$\neg(A \times B)$	$\neg A$	$\neg B$	$\neg A + \neg B$
0	0					
1	0					
0	1					
1	1					

Problem 8 *Negate the following statement.* **My dad likes to watch football and baseball.**

The following two formulae, known as *De Morgan's laws*, generalize the ones proven in Problems 5 and 7.

$$\neg(A_1 + A_2 + \ldots + A_n) = \neg A_1 \times \neg A_2 \times \ldots \times \neg A_n \qquad (1)$$

$$\neg(A_1 \times A_2 \times \ldots \times A_n) = \neg A_1 + \neg A_2 + \ldots + \neg A_n \qquad (2)$$

Problem 9 *Simplify the following formulae so that they do not contain a negation of a composite statement.*

- $X = \neg(AB) + \neg B$

$X =$

- $Y = \neg(\neg BC + C)$

$Y =$

- $Z = \neg(\neg AC) + B\neg C$

$Z =$

Problem 10 *Given the statements*

A = *Bob is driving to work.*

B = *Bob is shaving.*

form the statement $X = \neg(AB) + \neg B$ *from Problem 9 in plain English and simplify it in the space below.*

Simplifying Boolean expressions

Two expressions of Boolean algebra are called *equivalent* if they are equal as functions – the same inputs produce the same outputs. The latter can be checked by means of the corresponding truth tables. For example, the expressions $A + BC$ and $(A + B)(A + C)$ are equivalent.

Problem 11 *Prove that the expressions $A + AB$ and A are equivalent without using the truth table.*

Then use the truth table to check your proof.

A	B	AB	$A + AB$
0	0		
1	0		
0	1		
1	1		

In Problem 11, we have proven the following remarkable Boolean algebra equivalence.

$$A + AB = A \tag{3}$$

Here is one more.

Problem 12 *Prove the following formula without using the truth table.*

$$A(A + B) = A \tag{4}$$

To *simplify* a Boolean algebra expression means to find an equivalent expression that

1. contains no negations of composite statements; and

2. has as few simple statements as possible.

The equivalence

$$A(A + \neg(VW + \neg XYZ)) = A \tag{5}$$

is an example of such a simplification.

Problem 13 *If you decide to check formula 5 using a truth table, how many different inputs would you need to consider?*

Problem 14 *Prove formula 5.*

The following two problems present two more very important equivalences.

Problem 15 *Prove the following formula.*

$$AB + A\neg B = A \tag{6}$$

Problem 16 *Prove the following formula.*

$$(A + B)(A + \neg B) = A \tag{7}$$

Problem 17 *Simplify the following Boolean algebra expressions.*

- $\neg(AB) + \neg A =$

- $A + \neg(\neg AB) =$

- $\neg(\neg A \neg B) + \neg A =$

- $\neg(A + \neg(\neg AB)) =$

- $\neg(AB) + \neg ABC =$

The problem continues on the next page.

- $\neg A + \neg(AB + \neg B) =$

- $A\neg BC + A\neg(BC) + ABC + A\neg B =$

The problem below is similar to the logical problems you have solved at various math competitions. Please solve it any way you like. Further we will show you how to solve the problem using the Boolean algebra machinery we have developed.

Problem 18 *The year is 3025. Four kids got to the final tour of GMC8 (Galactic Math Olympiad for 8th graders), Nathan, Michelle, Laura, and Reinhardt. Some knowledgeable ORMC fans discussed their chances to win. One student thought that Nathan would take the first place and Michelle would take the second. Another student thought that Laura would take the silver while Reinhardt would end up the last of the four. The third student thought that Nathan would be second and Reinhardt third. When the results of the competition came out, it turned out that each of the ORMC students had made only one of the two* *predictions correct. Please find the places Nathan, Michelle, Laura, and Reinhardt got at GMC8-3014.*

A Boolean algebra solution to Problem 18.

The following are the simple statements from Problem 18.

- $N_1 =$ *Nathan takes the first place.*
- $M_2 =$ *Michelle takes the second place.*
- $L_2 =$ *Laura takes the second place.*
- $R_4 =$ *Reinhardt takes the fourth place.*
- $N_2 =$ *Nathan takes the second place.*
- $R_3 =$ *Reinhardt takes the third place.*

Let us use the simple statements above to translate the story into the Boolean algebra language. The first fan made a composite statement N_1M_2 that turned out to be false.

$$N_1M_2 = 0$$

The fact that a half of the guess is true means that either $N_1\neg M_2 = 1$ and $\neg N_1M_2 = 0$ or that $N_1\neg M_2 = 0$ and $\neg N_1M_2 = 1$. This can be expressed by means of a single formula.

$$N_1\neg M_2 + \neg N_1M_2 = 1 \tag{8}$$

A similar translation of the other two fans' predictions into the Boolean algebra language gives us the following.

$$L_2\neg R_4 + \neg L_2R_4 = 1 \tag{9}$$

$$N_2\neg R_3 + \neg N_2 R_3 = 1 \tag{10}$$

Multiplying 8, 9, and 10 brings together all the information we have about the competition.

$$(N_1\neg M_2+\neg N_1 M_2)(L_2\neg R_4+\neg L_2 R_4)(N_2\neg R_3+\neg N_2 R_3) = 1 \tag{11}$$

Let us first find the product of the second and third factors.

$$(L_2\neg R_4 + \neg L_2 R_4)(N_2\neg R_3 + \neg N_2 R_3) = 1$$

Opening parentheses gives the following.

$$L_2\neg R_4 N_2\neg R_3+L_2\neg R_4\neg N_2 R_3+\neg L_2 R_4 N_2\neg R_3+\neg L_2 R_4\neg N_2 R_3 = 1$$

Since Laura and Nathan cannot take the second place simultaneously, $L_2\neg R_4 N_2\neg R_3 = 0$. Since Reinhardt cannot take the third and fourth place at the same time, $\neg L_2 R_4\neg N_2 R_3 = 0$. The above sum shortens to just two terms.

$$L_2\neg R_4\neg N_2 R_3 + \neg L_2 R_4 N_2\neg R_3 = 1$$

This way, (11) boils down to the following:

$$(N_1\neg M_2 + \neg N_1 M_2)(L_2\neg R_4\neg N_2 R_3 + \neg L_2 R_4 N_2\neg R_3) = 1$$

Let us expand. $N_1\neg M_2 L_2\neg R_4\neg N_2 R_3 + N_1\neg M_2\neg L_2 R_4 N_2\neg R_3 + \neg N_1 M_2 L_2\neg R_4\neg N_2 R_3 + \neg N_1 M_2\neg L_2 R_4 N_2\neg R_3 = 1$ Since $N_1 N_2 = 0$, the second term is equal to zero. Since $M_2 L_2 = 0$, the third term is equal to zero as well. Since $M_2 N_2 = 0$, the same is true for the last term. We end up with the equation

$$N_1\neg M_2 L_2\neg R_4\neg N_2 R_3 = 1$$

that tells us the results of the competition. Nathan takes the first place, Laura the second, Reinhardt the third. Therefore, Michelle takes the fourth place. There are no contradictions: Michelle is not second, Reinhardt is not fourth, and Nathan is not second. We have solved the problem!

Problem 19 *Before the beginning of a school year, teachers get together to form a schedule. The math teacher wants to have her class either first or second. The history teacher wants to have his class either first or third. The English teacher wants to have her class either second or third. Please use Boolean algebra to help the teachers form the schedule. How many different possibilities do they have?*

ORMC old-timers have seen the following logical problems before. Now it's time to solve them again, with the help of Boolean algebra! To do so, give a name (assign a variable) to every simple statement in a problem. Then translate all the available information into some Boolean algebra formulae using the operations $+$, $\times$, and $\neg$. If possible, try to boil everything down to one formula. Simplify the formula and see if the simplified version makes the solution obvious.

Once upon a time in a land far far away, there lived a king who invented the following way of punishing criminals. Convicted lawbreakers were given a choice between two doors. Behind each door, there could be either a hungry tiger or a treasure of gold, but not nothing or both. The king would also post some warnings on the doors and then let the criminals choose.

Problem 20 *The king took the prisoner to the doors. There was a sign on each door. The first read, "There is gold in this room and there is a tiger in the other." The sign on the second door read, "There is gold in one of these rooms and in one of these rooms there is a tiger." "Are the signs true?" asked the prisoner. "One of them is," replied the king, "but the other is not. Now, make your choice, buddy!" Which door should the prisoner open? Why?*

Problem 21 *For the second prisoner, the following signs were put on the doors. Door 1: at least one of these rooms contains gold. Door 2: a tiger is in the other room. "Are the signs true?" asked the prisoner. "They are either both true or both false," replied the king. Which door should the prisoner choose? Why?*

Problem 22 *In this case, the king explained that, again, the signs were either both true or both false. Sign 1: either this room contains a tiger, or there is gold in the other room. Sign 2: there is gold in the other room. Does the first room contain gold or a tiger? What about the other room?*

Problem 23 *A prince travelling through a magic land found an enchanted castle guarded by an evil goblin. The goblin told the prince that he had a box with a key to the castle gate, but that it was very dangerous to open the box. The prince immediately accepted the challenge. The goblin presented the young man with three boxes, red, blue, and green. It was written on the red box, "Here is the key." The blue box read, "The green box is empty." The green box had a warning, "There is a poisonous snake in this box." "Ha-ha-ha", laughed the goblin, "it is true that one of these boxes has the key, one is home to a deadly snake and one is empty, but all the labels on the boxes lie. You can only try once. If you open the empty box, you go home empty-handed and if you open the box with the snake, you die!" Help the prince to choose wisely.*

Problem 24 *A says, "I am a boy". B says, "I am a girl". One of them is a boy while the other is a girl. At least one of them is lying. Who is a boy and who is a girl?*