Nimbers, II

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In this packet, we will dive deeper in the theory of impartial games. We will see that Nimber is not just a theory to determine who has a winning strategy, but also a practical guide to *execute* the winning strategy.

Let us sort out some definitions and rules.

1 Review

We use the set notation $G = \{G_1, G_2, \ldots, G_l\}$ to mean that $\{G_1, \ldots, G_l\}$ is the set of all positions you can legally move to if you start the game G. We say $G_1, \ldots, G_l$ are **immediate subgames** of G. The 0 game is formally defined as the empty set $0 = \{\}$, meaning that there is no legal move for the first player, and hence the first player loses.

1. Recall the nim game with a single pile of n tokens is denoted by $*n$. Write $*0$ and $*1$ in the set notation. Then write $*2$. *Hint.* What are all immediate subgames of $*2$? How do you write the positions themselves in the set notation? You will encounter sets of sets of ... of sets.

Throughout this packet, the set of natural number includes 0:

$$\mathbb{N} := \{0, 1, 2, \ldots\}.$$

Recall that the **Sprague–Grundy (SG) value** or the **nimber** of a game $G = \{G_1, G_2, \ldots, G_l\}$, which we denote today by $s(G)$, is defined as

$$s(G) := \min(\mathbb{N} \setminus \{s(G_1), \ldots, s(G_l)\}).$$

This operation that takes in a set of natural numbers and return the minimal natural number not in the set is called the **minimal exclusion**, and is denoted by

$$\operatorname{mex}\{n_1, \ldots, n_l\} := \min(\mathbb{N} \setminus \{n_1, \ldots, n_l\}).$$

2. Explain why the definition above implies that $s(0) = 0$.

3. Prove by induction that $*n$ has SG value n.

Recall that a game is in N-position if the first player has a winning strategy (from now on, we just say the first player **wins**), and P-position if the second player wins. The sum of two games G_1, G_2 is the game where we play G_1, G_2 side by side, and each player is allowed to take a move at exactly one of the games; the player who has no move left loses. Two games G_1, G_2 are called **equivalent** if $G_1 + G_2$ is in P-position, and we say $G_1 \approx G_2$.

4. Prove that a game G is in P-position if and only if $G \approx 0$. Consequently, two games G_1, G_2 are equivalent if and only if $G_1 + G_2 \approx 0$.

We need to prepare for an induction technique. Recall that games we consider are assumed to be **well-founded**, namely, there does not exist an infinite sequence of play $G = G_0 \to G_1 \to G_2 \to \dots$ such that G_{i+1} is an immediate subgame of G_i for all i (in other words, any given play of the game is guaranteed to finish in a finite numer of turns).

5. (Weak induction) Suppose P is a true-false statement that can be asked for all impartial games. Assume that whenever P is true for all immediate subgames of G then P is true for G.[1] Then P is true for all impartial games G unconditionally. *Hint.* If P is false for some impartial game G, then it is false for some immediate subgame G_1, and you can keep going forever.

We will actually need a stronger version of the induction principle. We say G' is a **proper subgame** of G if there is a sequence of play $G \to G_1 \to \cdots \to G_n$ with $n \geqslant 1$ such that $G' = G_n$. A **subgame** of G is either G itself or a proper subgame of G.

6. Explain why G cannot be a proper subgame of itself. *Hint.* Infinite chain.

7. (Strong induction) Fix an integer $n \geqslant 1$. Suppose P is a true-false statement that can be asked for all n-tuples of impartial games $(G_1, \dots, G_n)$. Given $(G_1, \dots, G_n)$, and assume that whenever P is true for all tuples $(G'_1, \dots, G'_n)$ (excluding $(G_1, \dots, G_n)$ itself) such that G'_i is a subgame of G_i for all $1 \leqslant i \leqslant n$, then P is true for $(G_1, \dots, G_n)$. Then P is true for all tuples $(G_1, \dots, G_n)$ unconditionally.

[1] In particular, P is true for 0 because P is *vacuously* true for all immediate subgames of 0, since 0 has no immediate subgame.

Now we prove some fundamental properties of the SG value.

Problem 8. Prove that if two games have the same SG values, then they are equivalent. *Hint.* Suppose $s(G_1) = s(G_2) = n$. Proceed by strong induction on (G_1, G_2) (*not* on n). Show that no matter what the first player plays in $G_1 + G_2$, the second player has a winning strategy.

9. Explain why any game with SG value n is equivalent to $*n$.

10. Prove that two equivalent games have the same SG values. *Hint.* You can try to prove the contrapositive. You end up needing to prove that two nim games $*n$ and $*m$ with $m \neq n$ are not equivalent by displaying a winning strategy of the game $*n + *m$ for the first player.

11. Explain why any game equivalent to $*n$ has SG value n.

2 Xor

Let **xor**, or **exclusive or**, be the operator on natural numbers defined by "adding binaries without carrying". We denote xor by $\oplus$. For example,

$$6 \oplus 3 = (110)_2 \oplus (011)_2 = (101)_2 = 5.$$

12. Verify that $a \oplus b = b \oplus a$, $(a \oplus b) \oplus c = a \oplus (b \oplus c)$. Verify that $a \oplus b = 0$ if and only if $a = b$.

Theorem 1. *For two games G_1, G_2,*

$$s(G_1 + G_2) = s(G_1) \oplus s(G_2).$$

We carefully prove this theorem by going through a winning strategy of Nim.

13. Explain why Theorem 1 implies that in a multi-pile nim game $*n_1 + \cdots + *n_l$, the second player has a winning strategy if and only if $n_1 \oplus \ldots \oplus n_l = 0$.

14. Explain why in order to prove Theorem 1, it suffices to show that the second player wins in the game $*a + *b + *(a \oplus b)$ for any $a, b \in \mathbb{N}$.

15. Suppose $a \oplus b \oplus c \neq 0$. Show that you may replace one of the numbers (let's say a), by a smaller natural number $0 \leqslant a' < a$, such that the resulting triple has "total xor" equal to 0, namely, $a' \oplus b \oplus c = 0$.

Problem 16. Use induction to prove that the second player wins the game $*a + *b + *(a \oplus b)$, and give a winning strategy. This finishes the proof of Theorem 1.

3 Dawson's chess

Consider a game, denoted by $U(n)$ from now on, as follows. For each natural number n, consider a $1 \times n$ chess board, so there is a row of empty squares. Players take turn putting a piece on one of the empty squares that is *not adjacent* to any existing piece. Whoever cannot move first loses. We denote the SG value of $U(n)$ by $u(n)$.

17. Play the game $U(n)$ with your neighbor with $n = 0, 1, \ldots, 7, 8$. For each n in the list, who do you think has the winning strategy? Do you have a more general guess?

Problem 18. Write down a recursive algorithm in terms of mex and xor to compute $u(n)$. Then compute $u(n)$ for $n \leqslant 16$. Does it confirm your guess of the last question? *Hint.* After you play a move, it becomes the sum of two smaller games.

The algorithm can quickly generate a large table of $u(n)$, but as you have noticed, there is no easy pattern. As it turns out, the SG values of $U(n)$ are eventually periodic with period 34. The table of $u(n)$ is shown in Table 1.

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
0+	0	1	1	2	0	3	1	1	0	3	3	2	2	4	0	5	2
17+	2	3	3	0	1	1	3	0	2	1	1	0	4	5	2	7	4
34+	0	1	1	2	0	3	1	1	0	3	3	2	2	4	4	5	5
51+	2	3	3	0	1	1	3	0	2	1	1	0	4	5	3	7	4
68+	8	1	1	2	0	3	1	1	0	3	3	2	2	4	4	5	5
85+	9	3	3	0	1	1	3	0	2	1	1	0	4	5	3	7	4
$\cdots$																	

Table 1: The values of $u(n)$. The last two rows will repeat themselves indefinitely, showing that $u(n)$ is eventually periodic with period 34.

19. List all $n \leqslant 100$ such that the first player loses in $U(n)$.

For all n not in your list, you know there is *theoretically* a winning strategy for $U(n)$ if you are the first player. However, can you practically win the $U(n)$ game, assuming you can freely consult Table 1 during the play?

Problem 20. Describe a recipe to execute the winning strategy for $U(n)$ (if there is one) with the help of Table 1. Play the game with your neighbor with $n = 12$ using the table-assisted winning strategy. If you want more challenge, try $n = 18$. (Note that naïve analysis even for $n = 10$ already seems impossible.)

The game $U(n)$ is also called **Dawson's chess** after a chess puzzle themed game equivalent to it.

4 Octal games

We have seen that Sprague–Grundy value is a powerful tool to analyze impartial games, and it is robust to rule variations.

21. Consider the game **Kayles**, called $K(n)$, where there is a row of n bowling pins and at a player's turn, they can knock down one pin or two adjacent pins. The player who knocks down the last pin wins. Who has the winning strategy? Is there a similar algorithm like the one before to compute the SG value of $K(n)$?

We now define a framework that includes lots of games like this. The interesting feature is that the games defined will always come in an infinite sequence, just like $U(n)$ and $K(n)$.

An **octal game** is played on a heap of n tokens. At each move, the player can remove certain number of tokens on one of the current heaps, and (possibly) split that heap into two. The player who cannot move lose. Whether a move is allowed will depend on the actual game. In an octal game, the allowed moves are determined by the number of tokens *removed*. For each $i \geqslant 1$, we use an octal digit $d_i \in \{0, 1, \dots, 7\}$ to encode information of whether we are allowed to remove i tokens from a heap as a move, and how many heaps are allowed to be left. More precisely,

$$d_i = a_i + 2b_i + 4c_i,$$

where

- $a_i = 1$ if taking all i tokens (and thus leaving zero heap) is allowed and zero otherwise,

- $b_i = 1$ if taking i tokens from a heap (leaving one heap) is allowed, and zero otherwise,
- $c_i = 1$ if taking i tokens from a heap and then dividing the remaining into two heaps is allowed, and zero otherwise.

The octal number $0.d_1d_2\dots$ is called the **octal code**. Note that a single octal code defines a sequence $G(n)$ of games for each $n \geqslant 0$.

For example, Nim is the octal game with code $0.333\dots$, becuase when we are handed a pile of length i, we are allowed to remove all of it (leaving 0 pile), or remove part of it (leaving 1 pile). This rule translates to $d_i = 3$ for all i.

22. Describe the rule of the game $0.07 = 0.07000\dots$ and 0.77 with n tokens. Is Kayles equivalent to one of these?

Problem 23. Find the octal game equivalent to the aforementioned $U(n)$, in terms of the octal code.

Any octal game defines a sequence of game $G(n)$, which in turn gives a nim sequence $g(n)$ given by the SG value of $G(n)$. Richard Guy conjectures that for any finite octal game (i.e., one whose octal code has digits eventually zero), the nim sequence $g(n)$ is eventually periodic. This is a remarkable open problem. There are examples that initially do not seem periodic, but it turns out to be, e.g., the nim sequence of the octal game 0.106 is eventually periodic with period 328226140474.

5 Multiplication of games

In the last package, you were introduced the rule for multiplying nimbers by a "rectangular grid" game. (If you haven't got to this part yet, don't worry! We won't assume knowledge of it as prerequisite.) But a question persists: why this particular game? Here we make it more natural by describing how we multiply two games in general.

Given two games G_1, G_2, define the product game $G_1 \times G_2$ as follows. To make a legal move as the first player of $G_1 \times G_2$, you must *simultaneously* declare a legal move in G_1 and a legal move in G_2. (If you cannot do so, you lose!)

The definition of $G_1 \times G_2$ is not really finished, but you can already answer the following question.

24. Explain why $G_1 \times 0 = 0 \times G_2 = 0$.

Now we shall describe the game state you leave to your opponent after you make your initial move. To set up the notation to keep track, let's say $G_1 = \{G_1', \ldots\}$ and $G_2 = \{G_2', \ldots\}$, so this means G_1' is a possible game state after you make a legal move in G_1, among others. Let's say in the game $G_1 \times G_2$, as your first move, you declare to play the move G_1' in G_1, and the move G_2' in G_2. We shall denote this move by a pair (G_1', G_2').

25. How many possible first moves can you make in the game $*2 \times *3$? List them all.

It remains to describe the game state that results from a move (G_1', G_2') in the game $G_1 \times G_2$. We define the resulting game state to be

$$G_1' \times G_2 + G_1 \times G_2' + G_1' \times G_2'.$$

In summary, we define $G_1 \times G_2$ to be, in set notation,

$$G_1 \times G_2 = \{(G_1', G_2') : G_1' \in G_1, G_2' \in G_2\},$$

and each (G_1', G_2') is a game defined by

$$(G_1', G_2') = G_1' \times G_2 + G_1 \times G_2' + G_1' \times G_2'. \tag{$*$}$$

This definition is inductive because it assumes the products of subgames have been defined.

26. Compute the SG value of $*2 \times *2$.

I haven't really convinced you why $(*)$ is a natural definition. It would be, if we can prove that

(a) (Commutativity) $G_1 \times G_2 \approx G_2 \times G_1$.

(b) (Distributivity) $(G_1 + G_2) \times G_3 \approx G_1 \times G_3 + G_2 \times G_3$.

(c) (Associativity) $(G_1 \times G_2) \times G_3 \approx G_1 \times (G_2 \times G_3)$,

and have a "natural" interpretation of both sides of each identity.

27. Explain why $G_1 \times G_2 \approx G_2 \times G_1$ is true in one or two sentences.

Now we work on distributivity. First let's review the rule of the game $G_1 + G_2$ in more depth.

28. In the game $G_1 + G_2$, suppose you make a move (i, G'_i), where $i \in \{1, 2\}$ and $G'_i \in G_i$. What is the resulting game state? *Hint.* It should be $G'_i + G_{3-i}$.

29. Show that $(G_1 + G_2) \times G_3$ has the same set of first moves as $G_1 \times G_3 + G_2 \times G_3$. *Hint.* A good notation to denote the first move in either game is by a triple (i, G'_i, G'_3), where $i \in \{1, 2\}, G'_i \in G_i, G'_3 \in G_3$. Interpret the meaning of this triple in the context of both games, carefully.

Problem 30. Prove the distributivity law. *Hint.* Write down the game state after the move (i, G'_i, G'_3) in $(G_1 + G_2) \times G_3$ as well as in $G_1 \times G_3 + G_2 \times G_3$. Show that they are equivalent. You may assume the distributivity law holds for subgames by strong induction. You may assume $i = 1$ to ease the notation. You will see the sum of six terms is equal the sum of four terms after some cancellation.

Now we work on associativity.

31. Show that the $(G_1 \times G_2) \times G_3$ has the same set of first moves as $G_1 \times (G_2 \times G_3)$. *Hint.* You actually only need to work on one side, say $(G_1 \times G_2) \times G_3$. If you get a description of the set of moves of $(G_1 \times G_2) \times G_3$ that is "unaware" of the parenthesis position in $(G_1 \times G_2) \times G_3$, then you are done.

Problem 32. In the game $(G_1 \times G_2) \times G_3$, describe the game state after the first move, assuming your first move is (G'_1, G'_2, G'_3) (the meaning of this should be clear if you have answered the previous question). Use distributivity to simplify your result. Then prove the associativity law.

Associativity law implies that the notation $G_1 \times \cdots \times G_l$ makes sense without specifying the parentheses.

Problem 33. Give a natural direct definition of the game $G_1 \times \cdots \times G_l$. Explain why being able to define $G_1 \times \cdots \times G_l$ directly gives a natural proof of the associativity law.

6 Additional problems

Problem 34. Analyze the octal game 0.07, namely, compute a table of SG values for $n \leqslant 16$ by first giving a recursive algorithm. (The game 0.07 is called **Dawson's Kayles**, and it is not the traditional Kayles.)

It was observed by Dawson that the game 0.07 and $U(n)$ are almost equivalent, up to a shift. Find the correspondence rule by matching the tables. Can you explain this coincidence? Even if not, do you have a strategy to prove that the correspondence holds for general n?

Problem 35. Prove that

(a) For a game G, we have $*1 \times G \approx G$.

(b) For a game G and a natural number n, we have

$$*n \times G \approx \underbrace{G + G + \cdots + G}_{n}.$$

(c) For games G_1, G_2, we have

$$(G_1 + G_2)^2 = G_1^2 + G_2^2.$$

(Here G^2 means $G \times G$.)

Problem 36. Show that if $a, b > 0$, then $*a \times *b \not\approx 0$.

Problem 37. Recall $2^{2^n} := 2^{(2^n)}$. Prove by induction on N the following statements:

(a) If $m < n < N$, then $*2^{2^m} \times *2^{2^n} \approx *(2^{2^m}2^{2^n})$.

(b) $*2^{2^n} \times *2^{2^n} \approx *\left(\frac{3}{2} \cdot 2^{2^n}\right)$.

(c) The set $\{0, 1, \ldots, 2^{2^N} - 1\}$ is closed under multiplication.

Conclude that $\{*0, *1, \ldots, *(2^{2^N} - 1)\}$ is a finite field with 2^{2^N} elements, namely, it is closed under addition, multiplication, and for any $*a \neq \{*1, \ldots, *(2^{2^N} - 1)\}$ there is a unique $*b \in \{*1, \ldots, *(2^{2^N} - 1)\}$ such that $*a \times *b \approx *1$. We denote $*b$ by $(*a)^{-1}$.

Problem 38. How would you compute in general $*a \times *b$? How about $(*a)^{-1}$?

Problem 39. We define the **Frobenius** of a game G simply by

$$F(G) := G^2.$$

For a positive integer n, define $F^n(G) = F(F^{n-1}(G))$ inductively, i.e., taking the Frobenius n times. Show that for a natural number a, we have $F^n(*a) \approx *a$ if and only if $a \in \{*0, *1, \ldots, *(2^{2^n} - 1)\}$.