Introduction to inequalities

Terry Wang

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The content of this packet will aim to cover some inequalities appearing in contest math. There will be two sections, first on algebraic and second on geometric inequalities; each section will begin with the easier inequalities and progress to the more difficult. You can start the second section without finishing the first. Problems marked * are more difficult/challenge problems.

1 Algebraic inequalities

The simplest inequality of course is $x^2 \geq 0$ for any real number x, as you'll find, however, this inequality is often exactly we will need in our subsequent proofs.

Problem 1 *Find all real numbers such that $x^2 = 0$, ie. when the inequality above is equality.*

Problem 2 *Prove the following inequality, for any $a, b \geq 0$ real numbers:*

$$a + b \geq 2\sqrt{ab}$$

Further show that equality occurs if and only if $a = b$.

Problem 3 *Prove the following inequality, for any $a, b \geq 0 real numbers:$*

$$2a^2 + 2b^2 \geq (a + b)^2 \geq a^2 + b^2$$

Further show that equality occurs if and only if $a = 0$ or $b = 0$.

Problem 4 *Prove the following inequality, for any a, b, c real numbers:*

$$a^2 + b^2 + c^2 \geq ab + bc + ca$$

Further show that equality occurs if and only if $a = b = c$.

Problem 5 ***(Cauchy-Schwarz inequality)*** *Let $u_1, \ldots, u_n$ and $v_1, \ldots, v_n$ be real numbers, then:*

$$\left(\sum_{i=1}^{n} u_i^2\right)\left(\sum_{i=1}^{n} v_i^2\right) \geq \left(\sum_{i=1}^{n} u_i v_i\right)^2$$

Another important observation that often helps is the fact that if $x \geq 0$ and $0 \leq y \leq 1$ then $xy \leq x$. For instance, consider the inequality:

$$a + b + c + d \geq ad + bc + cb + da$$

Where $0 \leq a, b, c, d \leq 1$. To prove this, notice that $0 \leq d \leq 1$, and so $ad \leq a$. Likewise we can reason for $bc \leq b$, $cb \leq c$, and $da \leq d$. Adding these four inequalities gives the result that we want.

Problem 6 *Let $a_1, \ldots, a_n \geq 0$ be real numbers, prove that:*

$$a_1^2 + a_2^2 + \cdots + a_n^2 \leq a_1 + a_1^3 + a_2 + a_2^3 + \cdots + a_n + a_n^3$$

Then using Problem 2, show that this can in fact be improved to:

$$2(a_1^2 + a_2^2 + \cdots + a_n^2) \leq a_1 + a_1^3 + a_2 + a_2^3 + \cdots + a_n + a_n^3$$

To get further, we are going to need to following concept:

Definition 1 *Let $f : \mathbb{R} \to \mathbb{R}$ be a real valued function. f is called convex if for any number $t \in [0,1]$ and $x, y \in \mathbb{R}$:*

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y)$$

Problem 7 ***(Young's inequality)*** *Prove that:*

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}$$

For $a, b \geq 0$ real numbers and $p, q > 1$ also real numbers such that $\frac{1}{p} + \frac{1}{q} = 1$. Hint: $f(x) := -\ln(x)$ is convex.

The main tool that comes with this definition is the following theorem:

Theorem 1 ***(Jensen's inequality)*** *Let f be a convex function, let $t_1, \dots, t_n \in [0,1]$ be such that $t_1 + \cdots + t_n = 1$, let $x_1, \dots, x_n$ be real numbers then:*

$$f(t_1x_1 + \cdots + t_nx_n) \leq t_1f(x_1) + \cdots + t_nf(x_n)$$

There are a couple ways to prove this, but the most elementary method requires the method of induction, which isn't precisely the focus of this lesson. So as an optional exercise, you may attempt to prove this yourself, but otherwise we will move forward assuming this theorem.

Problem 8 *Prove the following inequality from information theory, let $p_1, \dots, p_n$ and $q_1, \dots, q_n$ be real numbers between 0 and 1, such that $p_1 + \cdots + p_n = 1 = q_1 + \cdots + q_n$:*

$$0 \geq \sum_{i=1}^{n} p_i \ln p_i \geq \sum_{i=1}^{n} p_i \ln q_i$$

Hint: $\ln(x+1) \leq x$.

Problem 9 ***(QM-AM-GM-HM)*** *Prove that for $x_1, \ldots, x_n$ positive numbers:*

$$\sqrt{\frac{x_1^2 + \cdots + x_n^2}{n}} \geq \frac{x_1 + \cdots + x_n}{n} \geq \sqrt[n]{x_1 \ldots x_n} \geq \frac{n}{\frac{1}{x_1} + \cdots + \frac{1}{x_n}}$$

Note that this generalizes problems 2 and 4. Hint: There are three inequalities here, apply the Cauchy-Schwarz inequality for the first (left-most), apply Jensen's inequality with a particular choice of convex function for the second, reuse the second inequality for the third.

Problem 10 ***(Holder's inequality)*** *Let $x_1, \ldots, x_n, y_1, \ldots, y_n \geq 0$, and $p, q > 1$ be such that $\frac{1}{p} + \frac{1}{q} = 1$. Prove:*

$$\sum_{i=1}^{n} x_i y_i \leq \left(\sum_{i=1}^{n} x_i^p \right)^{\frac{1}{p}} \left(\sum_{i=1}^{n} y_i^q \right)^{\frac{1}{q}}$$

Hint: Use Young's inequality

Problem 11 ***(Rearrangement inequality)*** *Let $a_1 \leq \cdots \leq a_n$ and $b_1 \leq \cdots \leq b_n$ be sequences of non-decreasing real numbers, let $\pi : \{1, \ldots, n\} \to \{1, \ldots, n\}$ be a permutation of the numbers 1 through n. Prove that:*

$$a_1 b_1 + \cdots + a_n b_n \geq a_1 b_{\pi(1)} + \cdots + a_n b_{\pi(n)} \geq a_1 b_n + \cdots + a_n b_1$$

Problem 12 ***(Sharpiro's inequality, weak form)**** *Let $x_1, \ldots, x_n > 0$, prove:*

$$\frac{x_1}{x_2 + x_3} + \frac{x_2}{x_3 + x_4} + \frac{x_3}{x_4 + x_5} + \cdots + \frac{x_{n-2}}{x_{n-1} + x_n} + \frac{x_{n-1}}{x_n + x_1} + \frac{x_n}{x_1 + x_2} \geq \frac{n}{4}$$

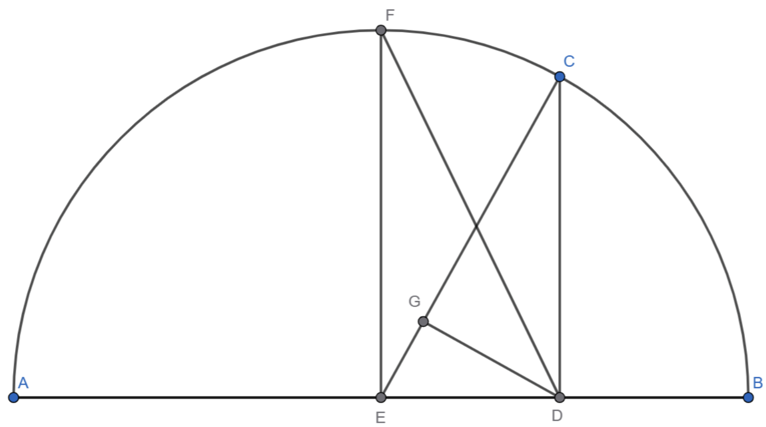
The right hand side can in fact be improved to $\frac{n}{2}$ if n is even and ≤ 12 or odd and ≤ 23 (you don't have to prove this), for larger n this is true with the right hand side $\gamma \frac{n}{2}$ where $\gamma \sim 0.9891$.

2 Geometric inequalities

Geometric considerations can often help in proving inequalities that have few variables but in complicated expressions, for instance, for non-negative numbers a, b:

$$\sqrt{\frac{a^2+b^2}{2}} \leq \frac{a+b}{2} \leq \sqrt{ab} \leq \frac{2}{\frac{1}{a}+\frac{1}{b}}$$

You may recognize this to be the QM-AM-GM-HM inequality in two variables, which of course required a fairly difficult proof in the general case. However, in this particular case we have an easier proof, consider the following diagram.



Problem 13 *Let $\overline{AD} = a$ and $\overline{DB} = b$. Now prove the following statements:*

- $\sqrt{\frac{a^2+b^2}{2}} = \overline{DF}$
- $\frac{a+b}{2} = \overline{EF}$
- $\sqrt{ab} = \overline{DC}$
- $\frac{2}{\frac{1}{a}+\frac{1}{b}} = \overline{GC}$
- *From the above 4 statements, conclude the AM-GM inequality in 2 variables.*

Problem 14 *Consider now a right triangle (which has two legs and a hypotenuse), which one of the three sides is the longest?*

Problem 15 ***(Triangle inequality)*** *Consider a triangle with side lengths a, b, c, is it possible for $a+b < c$? If so, give an example, if not, ie. $a+b \geq c$ always, then prove it. Hint: Use the fact that the shortest distance between two points is a straight line.*

Problem 16 *Let C be a circle of diameter d, let P, Q be points on the circle, prove that the distance between P and Q is smaller than d.*

Problem 17 *Let θ be an angle in a right triangle with corners A, B, C. Suppose that the right triangle is at A and that $\theta = \angle ABC$. Prove that if θ is expressed in units radians (recall that 180 degrees is π radians), that $\theta \geq \sin(\theta) = \frac{\overline{BC}}{\overline{BA}}$.*

Often it is hard to prove convexity for all values of t, here is a lemma that helps simplify this. We won't be going over the proof of this since it requires some analysis:

Lemma 1 *Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function (ie. a function that one can draw without picking up the pen, examples include polynomials, the exponential function, and $\ln$). If x, y are real numbers then:*

$$\frac{1}{2}f(x) + \frac{1}{2}f(y) \geq f\left(\frac{x+y}{2}\right)$$

Then f is convex, so when checking convexity, it suffices to check with $t = \frac{1}{2}$.

Problem 18 *Prove that $f(x) := -\sin(x)$ is convex for $x \in [0, \pi]$, here we take the convention that x is in units radians (180 degrees is π radians).*

Problem 19 ***(Weitzenböck's inequality)*** *Let a triangle have area A and side lengths a, b, c. Prove that:*

$$a^2 + b^2 + c^2 \geq 4\sqrt{3}A$$

Hint: Recall Problem 4 and Heron's formula for the area of a triangle, let $s := \frac{1}{2}(a + b + c)$ be half of the perimeter of the triangle, then:

$$A = \sqrt{s(s-a)(s-b)(s-c)}$$

Problem 20 ***(Ptolemy's inequality)*** *Let A, B, C, D be the corners of a quadrilateral, prove:*

$$\overline{AB} \cdot \overline{CD} + \overline{CB} \cdot \overline{AD} \geq \overline{AC} \cdot \overline{DB}$$

Problem 21 * *Let $x_1, \ldots, x_n$ be n points on the unit circle, let $P(x_1, \ldots, x_n)$ be the perimeter of the resulting polygon. Prove that:*

$$P(x_1, \ldots, x_n) \leq Q_n$$

Where Q_n is perimeter of the regular n-gon with points on the unit circle. Hint: Problem 18.

Problem 22 * *Let P be a polygon, prove that either P is a convex (ie. all interior angles are < 180), or that it is contained in a convex polygon with strictly smaller perimeter.*

*Now give an example to show that this is **not** true in 3D, ie. give an example of a non-convex 3D shape Q such that every convex shape which contains Q has strictly larger surface area.*

As a side note, convex in 3D is something that we can intuitively identify, but rigorously, a shape (2D or 3D) is convex if every line segment connecting two points within the shape is also contained in the shape itself.