Rep-Tiles and Other Tiling Problems

UCLA Math Circle Advanced 1B

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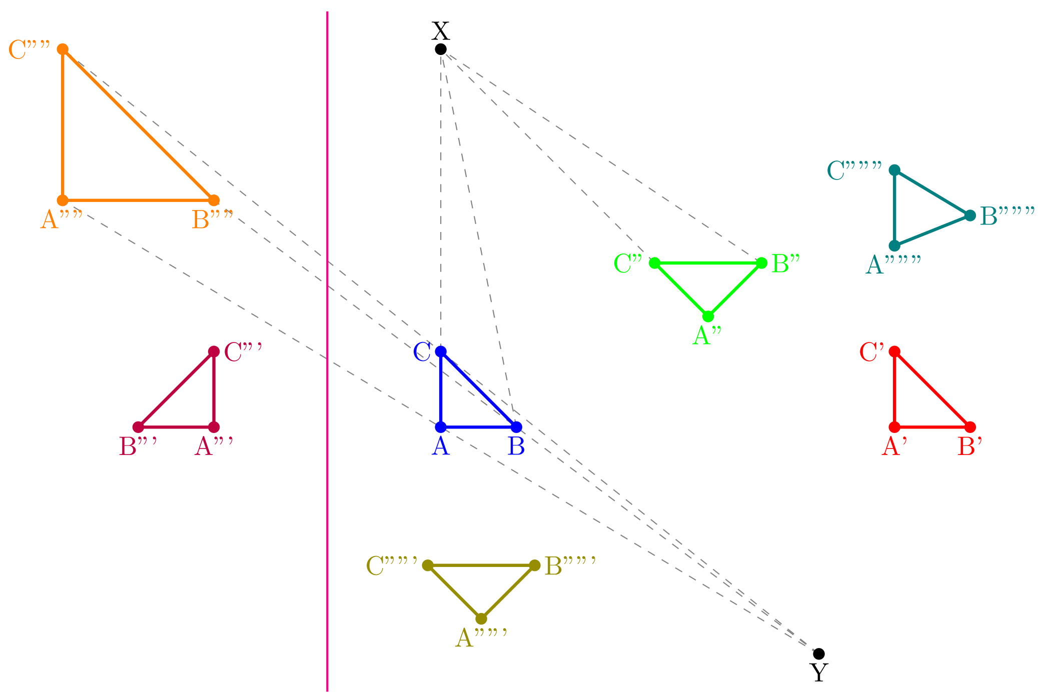
Introduction

Introduction to Tiling and Rep-tiles

Let's first introduce some geometric definitions. In 2-D Euclidean geometry, two objects are **similar** if one can be obtained from the other through translation, rotation, reflection and/or uniform scaling.

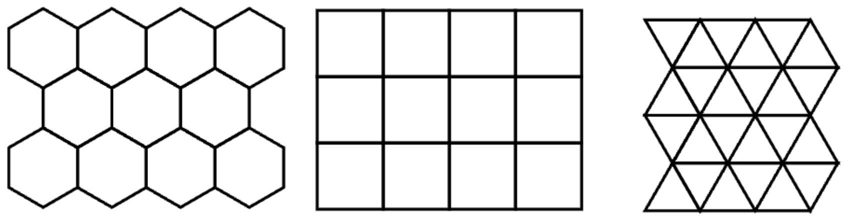
Problem 0.1 *Which of the triangles below is similar to the blue triangle ABC? Explain why. Which geometric operation(s) do we use for each similar triangle?*



However, two objects are **congruent** if one can be obtained from the other through translation, rotation and/or reflection (no uniform scaling!).

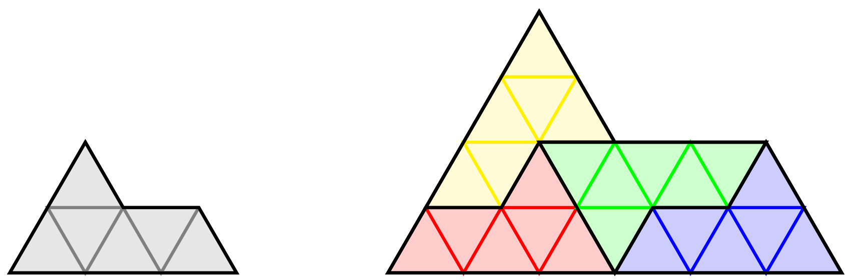
Problem 0.2 *Find an example of two triangles in the picture above that are similar but not congruent.*

In mathematics, a **tessellation** or **tiling** is the covering of a surface (which can be a plane or an object) by congruent "copies" of one or more objects. When discussing tessellations in math, we usually talk about coverings of a plane (but not for this packet). Below are tessellations of the plane by regular hexagons, squares, and triangles (we can't show the entire plane for each example, but we can see how the pattern for this tessellation continues infinitely in all directions):



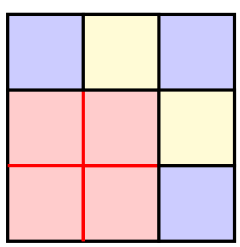
In this packet, we will talk about **rep-tiles** (not to be confused with the biological classification of animals). A rep-tile is a shape that can be partitioned (or **dissected**) into smaller "copies" which are similar to the original shape. In other words, these similar copies tile the object.

For example, 4 congruent "sphinx" shapes (each one formed using 6 equilateral triangles) form a large sphinx shape similar to each of its parts, so the sphinx shape is a rep-tile.



On the left is the sphinx tile. On the right is a scaled-up version of the object divided into 4 copies of the original tile (we call this a **dissection** of the rep-tile).

Rep-tiles do not always have to be dissected into congruent pieces, just similar ones. For example, we can dissect a square rep-tile into 6 copies the following way:



We label a rep-tile as **rep-n** if it uses n *congruent* copies in its dissection. If the dissection's copies aren't congruent, however, we label it as **irrep-n**. For example, the sphinx tile above is rep-4, but a square is irrep-6.

NOTE: For irrep-k dissections, the copies making up the original shape must not be congruent! Just because a rep-tile is rep-k doesn't necessarily mean it's irrep-k.

Problem 0.3 *How can we quickly tell if two polygons are similar, just by measuring their sides and angles? How can we quickly tell is two polygons are congruent?*

Problem 0.4 *Can create a tesselation of the 2D plane with the sphinx tile? What about any rep-tile?*

Problem 0.5 *Give another example of a 2D object that is a rep-tile and draw its dissection. Now give an example of an object that clearly isn't a rep-tile.*

Now let's look at some simple examples of rep-tiles and their properties:

1 Squares, Rectangles, and Parallelograms

Problem 1.1 *Is a square a rep-tile?*

Problem 1.2 *Prove that all squares are congruent, and use this to say that if a square is a rep-tile, then all squares are rep-tiles.*

Problem 1.3 *Can a square be rep-4? rep-6? rep-9?*

Problem 1.4 *For any square natural number k, prove that a square is rep-k. For this case, we call a square* ***rep-n^2****, where we assume $n \in \mathbf{Z}$ but we don't have to say so. Is a square rep-k for any non-square values of k?*

Problem 1.5 *Can a rectangle be a rep-tile? Is every rectangle a rep-tile?*

Problem 1.6 *Can two rectangle rep-tiles have different rep-n values? Given a rectangle rep-tile of length l and width w, find all possible values of k such that the rectangle is rep-k, with proof.*

Problem 1.7 *Find the length and width of a rectangle that is rep-2. Then find the lengths and widths of two non-similar rectangles that are both rep-15. Can you find a rep-n rectangle for any natural n?*

Problem 1.8 *Can a parallelogram be a rep-tile? Is every parallelogram a rep-tile?*

Problem 1.9 *Can two parallelogram rep-tiles have different rep-n values? Given a parallelogram rep-tile with sides of lengths a and b, find all possible values of k such that the rectangle is rep-k, with proof.*

2 Triangles

Unlike the previous shapes we've focused on, the properties of triangle rep-tiles are a bit more complicated.

Problem 2.1 *Prove that an equilateral triangle is rep-4. Then prove it is rep-n^2. Is it rep-k for any non-square values of k?*

Problem 2.2 *Prove that any triangle is rep-n^2.*

Problem 2.3 *Find a rep-4 triangle with 3 different dissections.*

Problem 2.4 *In general, how can we guarantee that a polygon rep-tile cannot be rep-k for some k? (Look back at why a square or an equilateral triangle can't be rep-2.)*

This observation can be very helpful in finding possible dimensions for a rep-k triangle:

Problem 2.5 *Using this knowledge, find a rep-2 triangle. Is there another non-similar triangle that is also rep-2? Repeat this problem for rep-3.*

Problem 2.6 *Use the previous problem to show that there exist rep-$2n^2$ and rep-$3n^2$ triangles. (The definitions of rep-$2n^2$ and rep-$3n^2$ are similar to the definition of rep-n^2.)*

Problem 2.7 *Find a rep-5 triangle. Is there another non-similar triangle that is also rep-5? Repeat this problem for rep-10.*

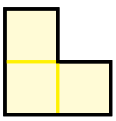
Problem 2.8 *Use the previous problem to show that triangles can be rep-k triangles where k is the sum of two squares.*

3 Polyominoes

For these next problems, we will cover rep-tile problems involving **polyominoes**, shapes formed by joining congruent squares edge-to-edge. We will also cover more general tiling problems as an added challenge.

Problem 3.1 *For what natural k do we know for sure that a polyomino is not rep-k?*

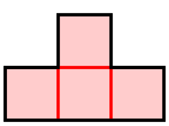
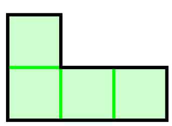
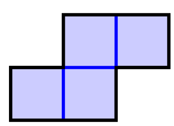
Problem 3.2 *For natural $k <= 6$, determine if the L-shaped **triomino** (a polyomino made from 3 squares) is rep-k^2:*



Then prove that this triomino is rep-2^n.

*** Challenge Problem 3.1** *For what natural n can a triomino formed from length 1 squares tile a length n square? For what n can tile a length n square missing one unit square in its corner?*

Problem 3.3 *For natural $k <= 6$, determine if each of the following **tetrominoes** (polyominoes each made from 4 squares) are rep-k^2:*

*** Challenge Problem 3.2** *For what natural n can each of the 3 tetrominoes formed from length 1 squares tile a length n square?*

* **Challenge Problem 3.3** *First, find irrep-7, irrep-8, and irrep-11 dissections of a square. (The definition of irrep-n is on page 2).*

Then prove that a square cannot be irrep-k for $k \in \mathbf{Z}$ *such that* $k \leq 5$. *(Hint: eliminate all "possible dissections" of the square.)*

Finally, prove that a square is irrep-k for any $k \in \mathbf{Z}$ *such that* $k \geq 6$. *(It may help to start with even k.)*

* **Challenge Problem 3.4** *Prove that any triangle is irrep-k for* $k \geq 6$*, but also give an example of a triangle that is irrep-n for some* $n < 6$.

* **Challenge Problem 3.5** *Prove that you cannot tile a length 8 square missing one unit square in its corner with 1-by-3 rectangles.*

Similarly prove that you cannot tile a length 10 square with 1-by-4 rectangles.

* **Challenge Problem 3.6** *Ninety nine 2-by-2 squares were cut out of a 29-by-29 board. Prove that it is always possible to cut at least one more 2-by-2 square.*