GRAPHS AND COLORINGS

A *graph* is an ordered pair $G = (V, E)$ comprising a set V of vertices or nodes together with a set E of edges or lines, which are 2-element subsets of V.

Colorings. A k-*coloring* of a graph G is a labelling of the graphs vertices with k colors such that no two vertices sharing the same edge have the same color. The smallest k such that a k-coloring exist is called the *chromatic number* of the graph, and denoted $\chi(G)$. The number of k-colorings of G is denoted $P_G(k)$. It turns out to be a polynomial in k, called *the chromatic polynomial* of G.

A graph is *planar* if it can be embedded in the plane, that is, it can be drawn on the plane in such a way that its edges intersect only at their endpoints. The Four Color Theorem says that the chromatic number of any planar graph is at most 4. It was proved by Appel and Haken in 1976, with the help of a computer.

1. The line graph L_n has n vertices numbered $1, \dots, n$, with edges between the vertices $i-1$ and i, for $2 \le i \le n$. Compute its chromatic number and polynomial.

2. The complete graph K_n had n vertices and an edge between any two distinct vertices. Compute its chromatic number and polynomial.

3. Let G be a graph and e an edge of G. We denote by $G - e$ the graph obtained from G by deleting e, and $G|e$ the graph obtained by suppressing e and identifying its two endpoints. Show that

$$P_G(k) = P_{G-e}(k) - P_{G|e}(k).$$

4. (uses induction) Show that $P_G(k)$ is a polynomial in k, for any graph G.

5. (uses induction) Suppose any vertex of G is connected by edges to at most k other vertices. Show that $\chi(G) \le k + 1$.

6. (uses induction) Suppose T is a tree with n vertices. Show that T has exactly $n - 1$ edges.

7. (uses induction) Suppose T is a tree with n vertices. Show that $P_T(k) = k(k-1)^{n-1}$.

8. Let C_n be the cycle graph with n vertices $1, \dots, n$, with edges between the vertices $i-1$ and i, for $2 \le i \le n$, and also an edge between n and 1. Compute its chromatic number and polynomial.

Planarity. Suppose G is a planar graph with v vertices, e edges, and f faces (including the infinite face). Euler's formula says that

$$v - e + f = 2.$$

9. Prove Euler's formula by induction on the number of faces.

10. Show that $e \le 3v - 6$ for any planar graph.

11. Suppose G is a planar graph that has at least three vertices and does not contain triangles, i.e. vertices a, b, c such that any two of them are connected by an edge. Show that $e \le 2v - 4$.

12. Show that the complete graph K_5 is not planar.

13. The complete bipartite graph $K_{p,q}$ has $p+q$ vertices of two types (p of type P and q of type Q) such that two vertices are connected by an edge if and only if they are of different types. Show that $K_{3,3}$ is not planar.

Kuratowski's theorem (1930) says that a finite graph is planar if and only if it does not contain a subgraph that is a subdivision of K_5 (the complete graph on five vertices) or $K_{3,3}$ (complete bipartite graph on six vertices, three of which connect to each of the other three). Here, a *subdivision* of a graph results from inserting vertices into edges (for example, changing an edge $\cdot - \cdot$ to $\cdot - \cdot - \cdot$) zero or more times.

Eulerian and Hamiltonian paths. An *Eulerian path* is a path in a graph which visits each edge exactly once. Similarly, an *Eulerian circuit* is an Eulerian path which starts and ends on the same vertex.

14. Show that if a graph has an Eulerian circuit, each of its vertices is connected to an even number of other vertices.

15. Find a similar restriction on the existence of an Eulerian path. Show that K_4 and $K_{3,3}$ do not have Eulerian paths.

A *Hamiltonian path* is a path in an undirected graph which visits each vertex exactly once. A *Hamiltonian cycle* is a cycle in an undirected graph which visits each vertex exactly once and also returns to the starting vertex.

16. Show that the complete graph K_n has exactly

$$\frac{1}{2}(n-1)!$$

Hamiltonian cycles.

17. Show that the bipartite complete graph $K_{n,n}$ has exactly

$$\frac{1}{2}(n-1)!n!$$

Hamiltonian cycles.