Sudoku

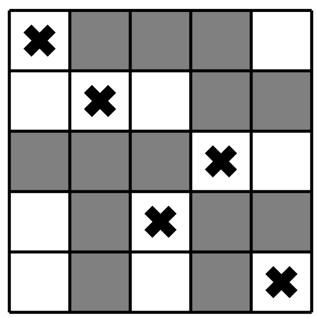
ORMC Olympiads

March 9, 2025

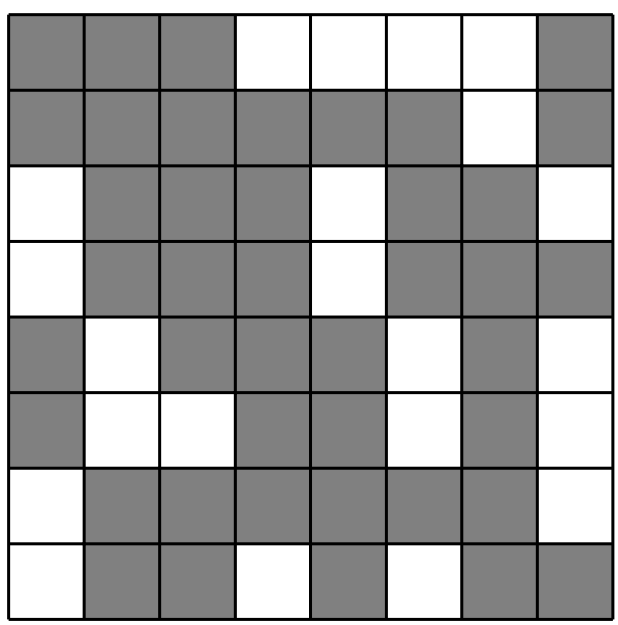
1 Crosses

You are given an $n \times n$ grid. Some of its squares are white, some are gray. Your goal is to place n crosses on white cells so that each row and each column contains exactly one cross.

Here is an example of such a grid, including a possible solution.



Problem 1. Find a solution for the following grid.



2 Place a digit

Problem 2. Place a digit

		5	4			6		2
		6		2		1	5	
2	9	3	5	6	1	7	8	4
	5	2	3		4	8		
3		1	2		6	4		5
				5	7	3	2	
	3			4	2	5	6	
	2	4			5	9		
5		7			9	2	4	

Hint: X-wing on nines

Problem 3. Place a digit

7		5	8		1	4	2	9
	2	8	9	4		1	7	
9	4	1	7		2			8
	8	3	1		7	2	9	4
1		4	2	9	3		8	7
2	7	9	4	8		3	1	
4		2	6	7	8	9		1
	9	7		1	4	8	6	2
8	1	6		2	9	7	4	

Hint: X-wing on fives

Problem 4. Place a digit

		3	9	1		7		
				2	3	4	9	1
1		9		4	7			6
	6	1	7					
		2	1		9	6		
				6	2		1	
7	9			8		1	2	3
3	1	8	2	9		5		
		5	3	7	1	9		

Hint: Find 2 X-wings in cols 3 and 4

Problem 5. Place a digit

9		8	7	3	5	1		
	1		9	8			3	
				2			9	8
8		5	4	6	9	3	1	
	9			7				
	4	3	2	5		9		
2	5			9				1
	8	9	5	1	2		6	3
		1	8	4	7			9

Hint: Swordfish on sixes

Problem 6. Place a digit

1			5		4	9	2	
	5			9		4		1
4	9	2		7		3		5
9	4							8
2	1	5		4		7	9	
6			9				4	2
		1	4	2		6		9
7		9		1				4
5		4			9		1	

Hint: Swordfish on sevens

Problem 7. Place a digit

3				4			5	
				3	7		4	8
						9	3	7
	1		4		3	5	8	
4				5			2	
9	5		2		8	4	7	
5			3				6	2
							9	5
6		9		2	5	3	1	4

Hint: Double swordfish, columns 2, 3, 4

3 Problems

Let $G = (X, Y, E)$ be a finite bipartite graph with parts X and Y and edge set E such that $|X| = |Y|$. A perfect matching a set of disjoint edges, which covers every vertex in X.

For a subset W of X, let $N_G(W)$ denote the neighborhood of W in G, the set of all vertices in Y that are adjacent to at least one element of W.

Theorem 8 (Hall's Marriage Theorem)**.** *There is a perfect matching in $G = (X, Y, E)$ if and only if for every subset W of X:*

$$|W| \le |N_G(W)|.$$

In other words, every subset W of X must have sufficiently many neighbors in Y.

Problem 9. Prove Birkhoff—von Neumann theorem: If a crosses puzzle has a fractional solution, it also has an integer solution.

Problem 10. Prove the original application of Hall's Marriage theorem: A *partial Latin square* is a $k \times n$ table with $k < n$ filled with numbers $1, \dots, n$ so that numbers in each row and column are different. Prove that one may continue any partial Latin square to an $n \times n$ Latin square.

Problem 11. Prove an alternative formulation of Hall's marriage theorem:

Let $\mathcal{F}$ be a finite family of sets. Define $X = \bigcup_{S \in \mathcal{F}} S$. A **transversal** of $\mathcal{F}$ is an injective function $f : \mathcal{F} \to X$ such that $f(S) \in S$ for all $S \in \mathcal{F}$. This is also called a *system of distinct representatives.*

The **marriage condition** states that for every subfamily $\mathcal{G} \subseteq \mathcal{F}$,

$$|\mathcal{G}| \le \Big| \bigcup_{S \in \mathcal{G}} S \Big|.$$

If a transversal exists, this condition must hold. Hall's theorem provides the converse:

Hall's Marriage Theorem: A finite family $\mathcal{F}$ has a transversal if and only if it satisfies the marriage condition.

Problem 12. Prove Ryser's theorem: if a crosses puzzle has a unique solution, then it can be solved using naked singles.