

## COMBINATORICS

MATH CIRCLE (ADVANCED) 1/13/2013

**Multiplication Principle:** if there are  $x$  ways of doing action A, and  $y$  ways of doing action B, then there are  $x \cdot y$  ways of performing both actions (i.e. performing action A **and** action B).

0) a) Suppose that John has four shirts, and three pairs of pants. How many different outfits does he have? List all the possible outfits.

12

b) Suppose John also has two pairs of shoes. How many different outfits does he have now? Draw a diagram to show your answer is correct.

24(draw a tree)

c) Suppose there are 5 people in a race. How many different ways are there to give out medals (gold, silver, and bronze) to the top 3 finishers?

$5 \cdot 4 \cdot 3$

d) How many different outcomes of the race are there?

$5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$

Recall that  $n! = 1 \cdot 2 \cdot 3 \cdots n$ .

**Permutations:** Suppose we have  $n$  objects, then there are  ${}_n P_k = \frac{n!}{(n-k)!}$  ways of choosing  $k$  of them where the order we choose the objects matters.

**Combinations:** Suppose we have  $n$  objects, then there are  ${}_n C_k = \binom{n}{k} = \frac{n!}{k!(n-k)!}$  ways of choosing  $k$  of them where the order we choose the objects does not matter.

1) Suppose there are 30 students in a class.

a) How many ways are there choose a group of 10 students from the class?

$\binom{30}{10}$

b) How many ways are there for 10 students to give presentations to the class?

$\frac{30!}{20!}$

c) How many ways are there to choose 10 students to be on the party planning committee and another 10 students to be on the field trip planning committee? Might there be more than one answer to this question?

$\binom{30}{10} \binom{20}{10}$  or  $\binom{30}{10} \binom{30}{10}$

d) How many ways are there to choose a  $P, VP, T, S$  for the class, as well as an advising committee of 8 students to help the officers?

$$\frac{30!}{26!} \cdot \binom{26}{8}$$

**Addition Principle:** if there are  $x$  ways of doing action A, and  $y$  ways of doing action B, then there are  $x + y$  ways of performing one of the actions (i.e. performing action A or action B).

2) a) Jane will either go to Subway or Burger King for lunch. Jane likes 6 different things at Subway and 4 different things at Burger King. How many different choices for lunch does she have?

$$6 + 4$$

b) After doing some shopping, John has four shirts, three pairs of pants, 5 pairs of shorts, two pairs of shoes, and one pair of boots. How many outfits does he have now?

$$4 \cdot (3 + 5) \cdot (2 + 1)$$

c) John is planning a weekend getaway. He leaves on Friday (wearing some clothes) and must pack his suitcase (with full outfits for two more days). How many ways can this be done (i.e. getting dressed on Friday and packing his suitcase)?

$$[4 \cdot 8 \cdot 3] \cdot \left[ \binom{3}{2} \cdot \binom{7}{2} \cdot \binom{2}{2} \right]$$

3) How many ways are there to rearrange the letters in:

a) *EAT*?

$$3!$$

b) *TEE*?

$$\frac{3!}{2!}$$

c) *EAT*, but not starting with *T*?

$$3! - 2!$$

**Overcounting 1 (Reverse Mult. Principle):** Suppose there are  $x$  ways to perform actions A and B. If there are  $y$  ways of performing action B, then there are  $x/y$  ways of performing action A.

**Overcounting 2 (Reverse Add. Principle):** Suppose there are  $x$  ways to perform action A or action B. If there are  $y$  ways of performing action B, then there are  $x - y$  ways of performing action A.

d) *UNCOPYRIGHTABLE*?

$$15!$$

e) *ANTIDISESTABLISHMENTARIANISM*?

$$\frac{28!}{4!3!5!1!4!2!1!1!1!2!1!}$$

f) *UNCOPYRIGHTABLE*, but not starting with *T*?

$$15! - 14!$$

4) Suppose there are 12 students in a class. We want to divide the class into groups to work on their homework. How many ways are there to do this if the groups are of size:

a) 8, 4

$$\binom{12}{8} \cdot \binom{4}{4}$$

b) 5, 4, 3

$$\binom{12}{5} \cdot \binom{7}{4} \cdot \binom{3}{3}$$

c) 6, 6

$$\frac{\binom{12}{6} \cdot \binom{6}{6}}{2!}$$

d) 4, 4, 4

$$\frac{\binom{12}{4} \cdot \binom{8}{4} \cdot \binom{4}{4}}{3!}$$

### ADDITIONAL PROBLEMS:

5) a) How ways are there to line up 5 people for a photograph?

$$5!$$

b) What if 2 people are identical twins?

$$\frac{5!}{2!}$$

c) What if instead, two people are a couple and must stand next to each other?

$$4! \cdot 2!$$

d) What if instead, two people are enemies and won't stand next to each other?

$$5! - 2! \cdot 4!$$

e) Repeat a) except in a circle which will go on a frisbee (i.e. the picture is from above).

$$\frac{5!}{5}$$

6) 20 people: Family of 4, 2 Families of 3, 3 couples, 4 singles.

a) How many ways are there to arrange the people in a line (in their groups)?

$$10! \cdot [4! \cdot (3!)^2 \cdot (2!)^3]$$

b) How many ways are there to arrange the people in a circle?

$$9! \cdot [4! \cdot (3!)^2 \cdot (2!)^3]$$

c) How many ways are there to arrange the people in a line so that the families of 3 are not next to each other?

$$[10! - 9! \cdot 2!] \cdot [4! \cdot (3!)^2 \cdot (2!)^3]$$

d)\* How many ways are there to arrange the people in a line so that no two singles are next to each other?

Hint: Place the singles first. This problem is just messy.

7) Suppose there are 12 students in a class. We want to divide the class into (indistinguishable) committees. A student can be on multiple committees. How many ways are there to do this if the committees are of size:

a) 8, 4

$$\binom{12}{8} \cdot \binom{12}{4}$$

b) 5, 4, 3

$$\binom{12}{5} \cdot \binom{12}{4} \cdot \binom{12}{3}$$

c)\* 6, 6

$$\binom{12}{6} + \frac{\binom{12}{6} \cdot \binom{12}{6} - \binom{12}{6}}{2!}$$

d)\* 4, 4, 4

$$\binom{12}{4} + \frac{\binom{12}{4}^2 - \binom{12}{4}}{2!} + \frac{\binom{12}{4}^3 - \binom{12}{4}^2}{3!}$$

**COMBINATORIAL PROOFS:**

A combinatorial proof is one that proves an identity  $A = B$  by showing that  $A$  and  $B$  count the same thing (in two different ways). Try to do all of the following using combinatorial proofs. Avoid any algebra if you can!

8) a)  $nP_k = \frac{n!}{(n-k)!}$ .

Hint: Count the number of ways to pick  $k$  people from  $n$  with order.

b)  $\binom{n}{k} = \frac{n!}{k!(n-k)!}$ .

Hint: Count the number of ways to pick  $k$  people from  $n$  without order.

c)  $\binom{n}{k} = \binom{n}{n-k}$

Hint: Either pick people for in the group, or to be excluded from the group.

d)  $\binom{n+1}{k+1} = \binom{n}{k} + \binom{n}{k+1}$ .

Hint: Suppose one of the  $k + 1$  people is Bob. Any group chosen from the  $k + 1$  people either has Bob or not.

e)  $2^n = \binom{n}{0} + \binom{n}{1} + \dots + \binom{n}{n}$ .

Hint: Count subsets of  $1, 2, \dots, n$  (remember that the empty set counts!). For the left hand side, each number can either be in the subset or not (2 choices).

f)  $\binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \binom{n}{3} + \dots \pm \binom{n}{n} = 0$ .

Hint: Rewrite as  $\binom{n}{0} + \binom{n}{2} + \dots = \binom{n}{1} + \binom{n}{3} + \dots$ . Consider Bob again.

g)  $\binom{n}{0}^2 + \binom{n}{1}^2 + \dots + \binom{n}{n}^2 = \binom{2n}{n}$ .

Hint: Suppose the  $2n$  people consist of  $n$  males and  $n$  females. Then use part c).**SOME COOL RESULTS:**

9) (A special case of Lucas' Theorem)

a) Show that  $k \binom{n}{k} = n \binom{n-1}{k-1}$ . See if you can find a combinatorial proof!Hint: Count the number of ways to choose a committee of  $k$  people as well as a leader for the committee (who is also on the committee) from  $n$  people.b) Suppose  $p$  is a prime,  $a \geq 1$ , and  $0 < k < p^a$ . Then

$$\binom{p^a}{k} \equiv 0 \pmod{p}.$$

Hint: Use part a). Recall that  $k < p^a$ ; what does this tell us about its prime factorization?

### 10) (Paths and Catalan Numbers)

Suppose that in moving in the  $xy$ -plane, one is only allowed to move up, i.e.  $+(0, 1)$ , or right, i.e.  $+(1, 0)$ . We will denote moving up by  $U$  and moving right by  $R$ .

a) Starting at  $(0, 0)$ , how many ways (i.e. **paths**) are there to get to  $(2, 2)$ ? to  $(5, 4)$ ? to  $(n, m)$ ?

There are  $\binom{n+m}{n}$  paths to  $(n, m)$ .

b) Prove that there are as many paths from  $(0, 0)$  to  $(n, m)$  as there are from  $(0, 0)$  to  $(m, n)$ ? Try to avoid an algebraic proof if you can!

An algebraic proof uses 8c). Otherwise reflect about the line  $y = x$ .

c) Suppose  $C_0 = 1$  and  $C_{n+1} = C_0 + C_1 + \cdots + C_n$ . What are  $C_1, C_2, \dots, C_5$ ?

See e).

d) Suppose  $C'_n = \frac{1}{n+1} \binom{2n}{n}$ . What are  $C_1, C_2, \dots, C_5$ ?

See e).

In what follows we outline a way to prove (combinatorially) that  $C_n = C'_n$  (called Catalan numbers).

e) How many paths are there from  $(0, 0)$  to  $(n, n)$  that do not pass above the diagonal ( $y = x$ ) for  $n = 1, 2, \dots, 5$ ?

There are 1, 2, 5, 14, 42 paths for  $n = 1, 2, 3, 4, 5$ .

f)\* Prove that there are  $C_n$  paths from  $(0, 0)$  to  $(n, n)$  that do not pass above the diagonal.

Hint: Look at where paths cross the line  $y = x - 1$ .

g)\* Prove that there are  $C'_n$  paths from  $(0, 0)$  to  $(n, n)$  that do not pass above the diagonal.

Hint: As part of the proof, you should prove that the number of paths from  $(0, 0)$  to  $(n, n)$  that do pass above the diagonal is equal to the number of paths from  $(0, 0)$  to  $(n - 1, n + 1)$ .