Lines and Planes in 3D

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1 Recall

In 3-dimensional space **the distance between two lines** is the smallest possible distance between two points on the lines (one on each line).

The **distance between point P and line q** is the smallest possible distance between P and a point Q in q.

Two lines p and q are then called **parallel** if for any point P we choose in p its distance from q remains constant.

Two lines that are neither parallel nor intersecting are called **skew lines**.

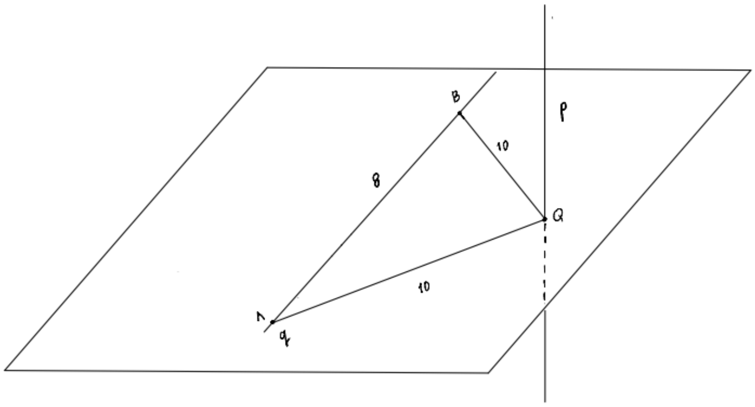
2 Recap

Problem 2.1.
Can two skew lines lie on the same plane?

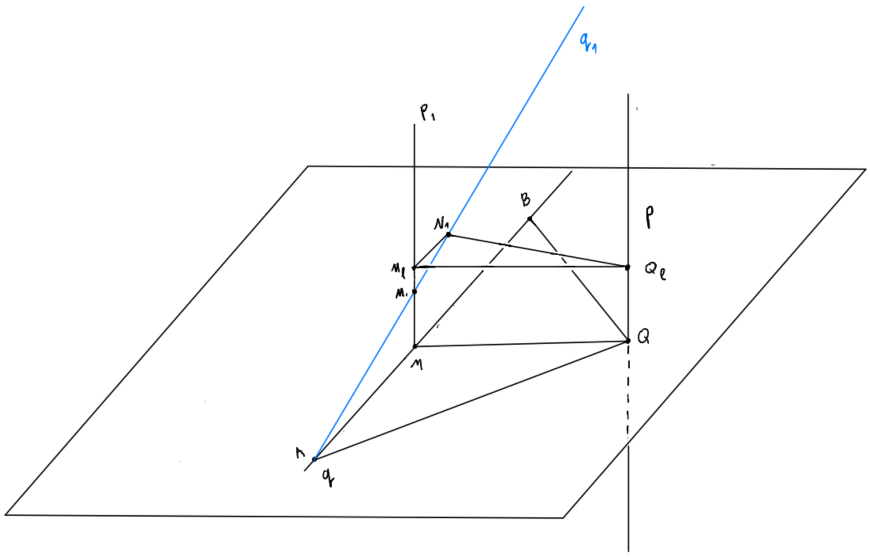
Problem 2.2.
Consider line p and point Q not on the line. If P is the point on the line that minimizes the line segment $\overline{PQ}$, what is the angle $\overline{PQ}$ forms with p?

Problem 2.3.
Distances between lines. Consider a line p and plane H perpendicular to p intersecting it at point Q.

- **Parallel:** Consider a line l parallel to p intersecting H at point L. If $\overline{LQ} = 6$ (*i.e. the length of the line segment* $\overline{LQ}$ is 6) what is the distance between p and l?
- **Perpendicular:** Consider a line q in the plane H. Consider points A and B on q such that $\overline{AQ} = \overline{BQ} = 10$, and $\overline{AB} = 8$. Find M on q that minimizes $\overline{MQ}$. What is the length of $\overline{MQ}$ and the angle $\angle QMA$?



- **Generally skew:** Pick a line q_1 such that its projection onto H is line q (q from the perpendicular case above). M_1 is the point on q_1 such that M is its projection onto H. Let p_1 be the extension of $\overline{MM_1}$.
  Pick an arbitrary point N_1 and consider a plane H' parallel to H passing through N_1. Let Q_l and M_l be the points where H' intersects p and p_1 respectively.



1. What is the angle $\angle N_1 Q_l Q$?
2. Show that $\overline{N_1 Q_l}$ is the smallest possible distance between N_1 and p.
3. Show that $\overline{Q_l M_l} = \overline{MQ}$ and that these segments are parallel.
4. What is the angle $\angle Q_l M_l N_1$?
5. Show that $\overline{Q_l M_l} < \overline{Q_l N_1}$.
6. Think how this means that the distance between q_1 and p is $\overline{QM}$.

Problem 2.4.
Given two skew lines p and q use the above to explain how to find a segment that is perpendicular to both of them with endpoints on these lines.

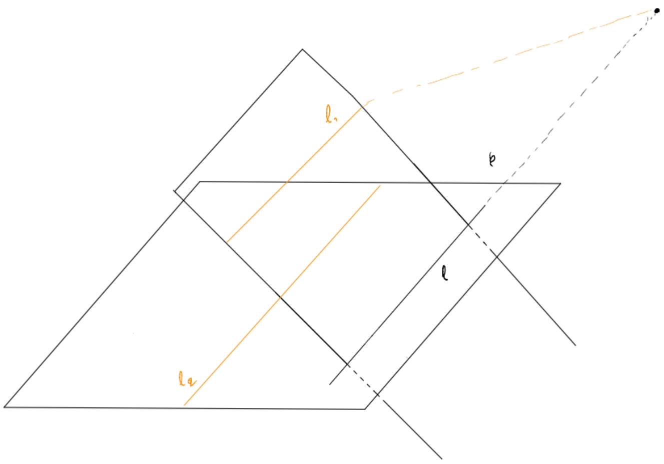
3 Planes and Parallels

Warmup:

Problem 3.1.
If 2 parallel lines have a common point are they the same line?

Problem 3.2.
Draw a line p that lies on a plane H. If q intersects the plane exactly once, can it be parallel to p?

Problem 3.3.
Assume l_1 and l_2 are parallel lines. Each one lies on a plane such that these two planes intersect along a line l. Is l_1 parallel to l or do they intersect?



We say a line l is parallel to a plane H if there is a line contained in H parallel to l.

Problem 3.4.
Consider two planes H_1 and H_2 that intersect along a line p. If l is parallel to both planes, show that it is parallel to p.

Problem 3.5.
Consider skew lines l and l_1 (as defined above).

1. Pick a point P_1 on l_1 and define the plane H that contains l and passes through P_1. Could l_1 lie on that plane?
2. How many planes can you find that contain l and intersect l_1 at a single point?
3. Pick two such planes H_1 and H_2; if some line l_2 is parallel to both planes, what is the angle between l_2 and l ?
4. Suppose l_2 is not parallel and doesn't intersect both l and l_1.
5. Suppose H_1 intersects l_1 at P_1 and l_2 at Q_1. Similarly, H_2 intersects l_1 at P_2 and l_2 at Q_2. Do lines $\overline{P_1Q_1}$ and $\overline{P_2Q_2}$ intersect l?
6. Do they intersect it at the same point?

Show you can find infinitely many lines each of which intersects all of the three lines.

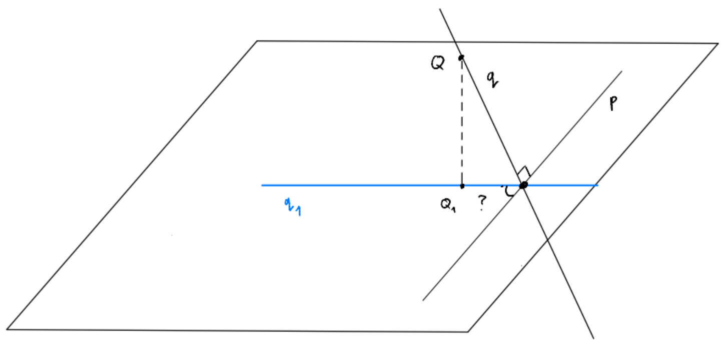
Problem 3.6.
Consider two intersecting lines l_1, l_2, and define H to be the plane containing both of them. Add a line l_3 that intersects both l_1 and l_2. Show that either they all meet at the same point or they all lie on the same plane. Can you generalize that for n lines where every two of them intersect?

4 Properties of Projections

Recall for a point B and a plane H the projection of B onto H is the point B' such that the segment $\overline{BB'}$ is perpendicular to the plane H (*i.e. perpendicular to all lines on* H).

Problem 4.1.
Consider two perpendicular lines p and q that intersect at a point O. Consider plane H such that p is contained in H and q is neither contained in it nor perpendicular to it. Consider also a plane J that is perpendicular to p and passes through O (*Note that plane* J *is not drawn in the figure below*).

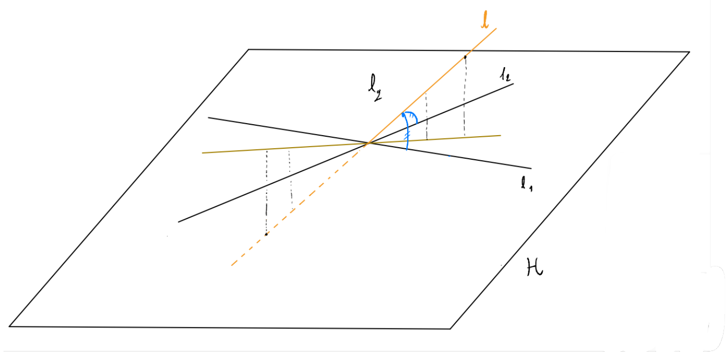
- Does the line q lie in the plane J?
- If Q is a point in q and Q_1 its projection onto H, is $\overline{QQ_1}$ perpendicular to p?
- Is $\overline{QQ_1}$ contained in J
- Show that the projection of q onto H is also perpendicular to p.



Problem 4.2.
Assume l, l_1, l_2 are lines and H is a plane such that:

1. l_1 and l_2 intersect and are contained in H.
2. l is not perpendicular to H.
3. l forms equal angles with l_1 and l_2.

Prove that the projection of l to the plane H also constitutes equal angles with lines l_1 and l_2.



Recall: a bisector of an angle $\angle AOB$ is the line containing points O and M such that $\angle AOM = \angle MOB$.

Problem 4.3.
Prove that line l forms equal angles with two intersecting lines if and only if it is perpendicular to one of the two bisectors of the angles between these lines.

Problem 4.4.
Prove that the line forming pairwise equal angles with three pairwise intersecting lines that lie in plane H is perpendicular to H.

5 Three Perpendiculars

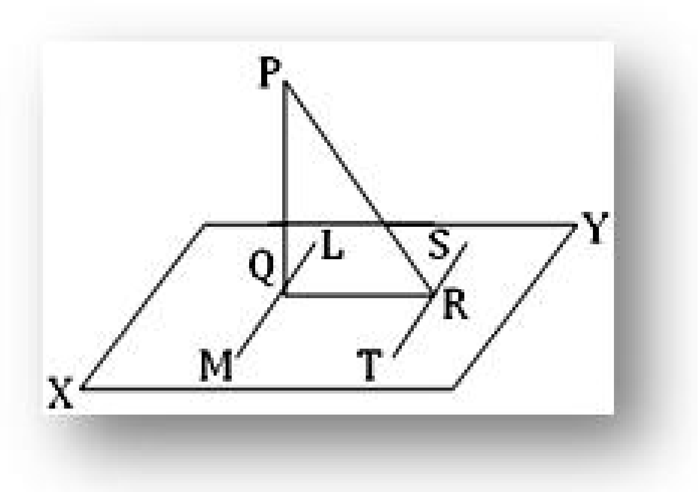
Problem 5.1.
Let p be a line not perpendicular to plane H. Let q be a line contained in H. Show that: p is perpendicular to q,

if and only if

the projection of p onto H is perpendicular to q.

The Theorem of Three Perpendiculars:

Theorem 1.
If $\overline{PQ}$ is perpendicular to a plane XY and if from Q, the foot of the perpendicular, a straight line $\overline{QR}$ is drawn perpendicular to any straight line $\overline{ST}$ in the plane, then $\overline{PR}$ is also perpendicular to $\overline{ST}$.



Problem 5.2.
Prove the above Theorem.

Problem 5.3.
Prove that the opposite edges of a regular tetrahedron are perpendicular.

Problem 5.4.
Edge $\overline{AD}$ of a tetrahedron $ABCD$ is perpendicular to face ABC. Prove that the projection to plane BCD maps the orthocenter of triangle ABC into the orthocenter of triangle BCD.

6 Hard-Problems

Problem 6.1.
Given two skew lines l_1 and l_2; points O_1 and A_1 are taken on l_1; points O_2 and A_2 are taken on l_2 so that O_1O_2 is the common perpendicular to lines l_1 and l_2 and line A_1A_2 forms equal angles with lines l_1 and l_2. Prove that $O_1A_1 = O_2A_2$.

Problem 6.2.
Points A_1 and A_2 belong to planes H_1 and H_2, respectively, and line l is the intersection line of H_1 and H_2. Prove that the line A_1A_2 forms equal angles with planes H_1 and H_2 if and only if points A_1 and A_2 are equidistant from line l.

Problem 6.3.
In space, there are given two skew lines l_1 and l_2 and point O not on any of them. Does there always exist a line passing through O and intersecting both given lines?