

# GRAPHS AND PARTIES

BEGINNER CIRCLE 11/11/2012

## 1. THE PARTY OF EVIL TWINS

Isaac and his evil twin, Saruman, decide to have a party at their house. Both of them are obsessed with handshaking. They invite the following people to their party:

- Jeff, and his sinister brother Fej,
- Morgan, and his devilish twin sister Morgana
- Jonathan, and his terrible twin brother Johnathan
- Derek, and his evil twin Devön

When everybody walks in, people shake hands. Of course, nobody shakes hands with their evil twin. While shaking hands, Isaac notices that Saruman, Jeff, Fej, Morgan, Morgana, Jonathan, Johnathan, Derek and Devön all shake a different number of hands.

**Problem 1.** What is the most hands that anybody could have shaken that night?

**Problem 2.** What is the fewest number of hands that anybody could have shaken that night?

**Problem 3.** What is the total number of handshakes that happened that night?

**Problem 4.** Draw a possible graph of the handshakes that occurred at the party

**Problem 5.** How many handshakes did Isaac have that evening?

**Problem 6.** How many handshakes did Saruman have that night?

## 2. A QUICK REVIEW

A **graph** is a collection of objects called **vertices** and a collection of **edges** that go in between them, which have the following properties:

- There is at most one edge connecting any two vertices
- Every edge connects two different vertices

Here are some examples of graphs and non-graphs

FIGURE 2.1. Things that are Graphs

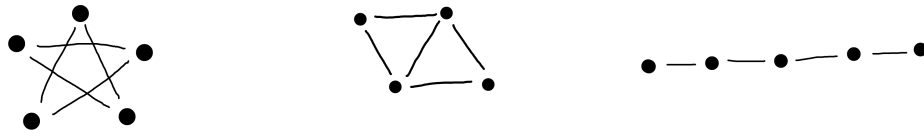
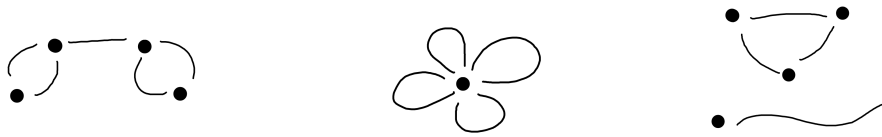


FIGURE 2.2. Things that are not Graphs



When we have a graph, one of the properties we are most interested in is the number of edges that connect to each vertex. If  $v$  is a vertex of a graph, then the **degree** of  $v$  (written  $\deg v$ ) is the number of edges that connect to that vertex. If no edges connect to the vertex, we say it has degree 0.

## 3. SOME NEW THINGS!

We call a graph **two parted** if its vertices form two clumps, and each clump only has edges going to the other clump.



When we talk about graphs, we don't really care about the position of the vertices, just

the edges that are between them. For instance, the following graphs are the same in our view:



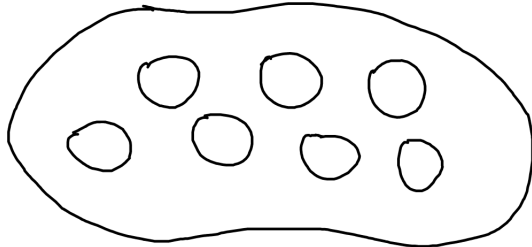
**Problem 7.** Remember, we proved last week that the sum of the degrees of vertices in a graph was twice the number of edges. Explain why this is true in a few sentences.

**Problem 8.** Using the previous problem, conclude that every graph has an even number of vertices with odd degree.

**Problem 9.** We say that a graph is connected if every two vertices can be connected with a path. Suppose that every vertex has even degree, and that the graph is connected. Show that if you remove one edge from the graph, it is still connected. (Hint: Show that if every two vertices are contained in a cycle.) Write your solution using full sentences.

**Problem 10** (From *Graphs* by Gurowitz and Khovring). On the first floor of the Math Science building, there is a giant lake with 7 islands in it. Every island has 1, 3 or 5 bridges, and the bridges may connect to the shore or between islands.

- (a) Can you put in bridges in the following picture so it satisfies the above conditions?



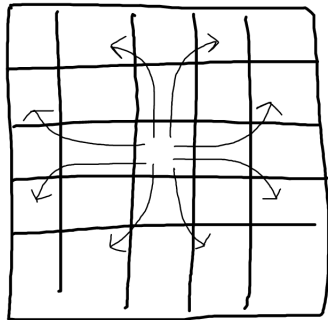
- (b) Is there a way of placing bridges so that they satisfy the above conditions, and no bridge connects to the shore? Explain why or why not in full sentences.

**Problem 11** (From *Graphs* by Gurowitz and Khovring). In the country of Numbers, there are cities numbered 1 through 9, and there are highways between every two cities. Each highway is labeled with the two digits of the cities that it connects. For example, the freeway from city 3 to city 5 is called, “The 35.”

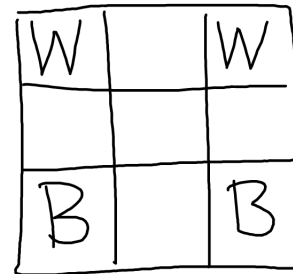
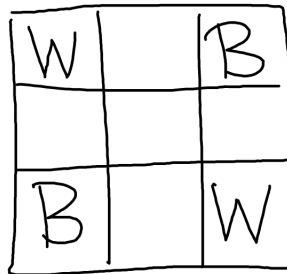
- (a) Jonathan enjoys the number 3, and will only drive on a highway if the highway number is divisible by 3. Is it possible for him to travel from City 1 to City 7? Explain why or why not in full sentences.

- (b) Morgan loves the number 9, and will only drive on a highway if the highway number is divisible by 9. Is it possible for him to travel from City 1 to City 9? Explain why or why not in full sentences.

A knight is a chess piece that can move in a  $L$  shape, like below



**Problem 12.** Can you move the knights from the position on the left to the position on the right without one of the knights taking the other?



(a) Create a graph where the vertices are the squares on a chess board, and two squares are connected if a knight can move from one square to the other.

- (b) Can you draw the same graph as a cycle and a single vertex?
- (c) Why does this graph being a cycle show that you can't move from the position on the left to the position on the right without one knight taking a different one? Explain using full sentences.
- (d) Can you draw that graph above as a two-parted graph?

**Problem 13.** The length of a cycle is the number of vertices in it. Can you show that if a graph is two parted, then every cycle has an even number of vertices in it? Use full sentences.

**Problem 14.** Can you draw a graph with 6 vertices, which is two parted, and has 6 edges?

**Problem 15.** Jeff wants to organize a tournament for a math circle combination at the end of the quarter. He wants to have 5 teams, and he wants each team to play in three contests against another team. Why is this a problem? Explain in full sentences. (Hint: Try drawing a graph)

**Problem 16.** Derek is designing a new set of corridors for Math Science which are to be safe in the case of an earthquake. Each corridor connects 2 rooms together. Every 2 rooms must be connected by a series of corridors and rooms, but the number of corridors that one has to travel through to get to any other room isn't so important. Here is an example of a corridor system that would work, and one that wouldn't:



What is a way to draw these corridor systems using graphs?



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- (a) Due to budget cuts, Derek is ordered to build a corridor system for 3 rooms that uses the smallest number of corridors. How does he do it?
- (b) Derek is ordered to build a corridor system for 5 rooms that uses the smallest number of corridors. Can you come up with two different floor plans that work?
- (c) Derek wants to design a corridor system for 5 rooms that is still usable if a single corridor collapses. How many corridors does he need to build?
- (d) The 5th floor of Derek's building has 7 rooms. What is the largest number of corridors that one would have to walk through to get from one room to another?
- (e) Due to a market boom in the corridor building business, Derek is instructed to make every room connected to every other room by a single corridor in a building with 5 rooms. How many corridors will Derek have to construct?

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(f) Can Derek build such a corridor system without having 2 corridors cross each other? Explain your reasoning in full sentences.

(g) Derek wants to build a corridor system for a floor with 4 rooms, such that if an earthquake takes out any 2 corridors, the corridor system still connects all the rooms together. How many corridors must he build?

**Problem 17.** Suppose that I have a graph with 4 vertices, and every vertex is connected to every other vertex. How many different paths are there from one vertex to another?

**Problem 18.** In this problem, we will show that the sum of

$$1 + 2 + 3 + \dots + n = n \times (n + 1) \div 2$$

Let us start by showing that

$$1 + 2 + 3 + 4 = 4 \times 5 \div 2$$

- (a) Draw a graph with 5 vertices.
- (b) Select an vertex, and then draw edges from that vertex to every other vertex. How many edges have you drawn?
- (c) Select another vertex, and then draw edges from that vertex to every other vertex. How many *new* edges have you drawn?
- (d) Select a third vertex, and then draw edges from that vertex to every other vertex. How many *new* edges have you drawn?
- (e) Repeating this process, how many edges do you end up drawing (represent this as a sum of the number of edges you drew on each step)

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- (f) What is the degree of each vertex?
- (g) As every vertex has the same degree, what is the sum of the degrees of the vertices?
- (h) How does this show the result that we want? Answer in full sentences.
- (i) Repeat the same process, except with  $n + 1$  starting vertices. Give a proof of the identity,

$$1 + 2 + 3 + \dots + n = n \times (n + 1) \div 2$$