# Lines and Planes in 3D

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## 1 Definitions

In 3-dimensional space **the distance between two lines** is the smallest possible distance between two points on the lines (one on each line).

The **distance between point** $P$ **and line** $q$ is the smallest possible distance between $P$ and a point $Q$ in $q$.

Two lines $p$ and $q$ are then called **parallel** if the distance between point $P$ in $p$ and $q$ is the same for any point on p.

**Problem 1.1.**
Draw two lines in 3D space that don't intersect but are not parallel.

Two such lines are called **skew lines** (i.e.if they are neither parallel nor intersecting).

For the rest of the worksheet you can think of a plane as a flat surface. A plane can be determined by the following.

1. 3 non-*collinear* points (i.e. points that don't all lie on one line).
2. A line and a point not on it.
3. Two parallel lines.
4. Two intersecting lines.
5. A line and a point $A$ on the line. *(Such plane would be the set of points $B$ such that the line segment $AB$ is perpendicular to the line).*

# 2 Angles and Distances

**Problem 2.1.**
Consider a line $p$ and plane $H$ perpendicular to $p$, intersecting $p$ at $A$. (*To help your visual intuition, think of line $p$ as a vertical line*). Take line $q$ in the plane. Show that the distance between $p$ and $q$ (lines) is the same as the distance between $A$ and $q$ (point and line).

**Problem 2.2.**
Consider a line $p$ and plane $H$ perpendicular to $p$, intersecting $p$ at $A$. Take line $q$ in the plane and let $B$ be the closest point between $A$ and $q$. Consider the plane $J$ that contains $q$ and is parallel to $p$.

- Show that all lines in plane $J$ passing through $B$ are equal distance away from $p$.
- Show that all lines in plane $J$ not parallel to $p$ are equal distance away from $p$.

The projection of a point $A$ on a plane $H$ is the point $B$ such that the segment $AB$ is perpendicular to the plane.

**Problem 2.3.**
Consider two skew lines $p$ and $q$. Draw the plane perpendicular to $p$. Show that the distance between $p$ and $q$ is the same as the distance between $p$ and the projection of $q$ in the plane.

**The result above is pivotal for finding distances between skew lines in 3D space.**

**Problem 2.4.**
Given cube $ABCDA_1B_1C_1D_1$ with side length $a$. Find the angle and the distance between lines $A_1B$ and $AC_1$.

**Problem 2.5.**
Given cube with side length 1. Find the angle and the distance between diagonals of the two of its neighboring faces.

**Problem 2.6.**
Given cube $ABCDA_1B_1C_1D_1$ with side length 1, let $K$ be the midpoint of edge $DD_1$. Find the angle and the distance between lines $CK$ and $A_1D$.

**HARD:**

**Problem 2.7.**
Edge $CD$ of tetrahedron $ADBC$ is perpendicular to plane $ABC$; $M$ is the midpoint of $DB$, $N$ is the midpoint of $AB$ and point $K$ divides edge $CD$ in relation $CK : KD = 1 : 2$. Prove that line $CN$ is equidistant from lines $AM$ and $BK$. *Hint: The intersection point of the medians of a triangle is $2/3$ of the distance from the vertex to the middle of the opposite side.*

**Problem 2.8.**
Consider 4 spheres of equal radius $r$. 3 of them are placed as close as possible to each other on the ground. The fourth rests on top of the other three (on the dent they form). What is the height of the structure from the ground as a function of $r$.

# 3 Lines and Planes

**Problem 3.1.**
Consider perpendicular lines $p$ and $q$. Pick a plane passing through $p$ such that it is not perpendicular to $q$ and $q$ is not contained in it. Let $q_1$ be the projection of $q$ onto that plane. Show that $p \perp q_1$. *Note that the projection of $q$ onto the plane is a line because it is not perpendicular to it.*

**Problem 3.2.**
Assume $l$, $l_1$, $l_2$ are lines and $H$ is a plane such that:

1. $l_1$ and $l_2$ intersect and are contained in $H$.
2. $l$ is not perpendicular to $H$.
3. $l$ forms equal angles with $l_1$ and $l_2$.

Prove that the projection of $l$ to the plane $H$ also constitutes equal angles with lines $l_1$ and $l_2$.

**Problem 3.3.**
Prove that line $l$ forms equal angles with two intersecting lines if and only if it is perpendicular to one of the two bisectors of the angles between these lines.

**Problem 3.4.**
Given two skew lines prove that there exists a unique segment perpendicular to them and with the endpoints on these lines. Show how to find it.

**HARD:**

**Problem 3.5.**
Given two skew lines $l_1$ and $l_2$; points $O_1$ and $A_1$ are taken on $l_1$; points $O_2$ and $A_2$ are taken on $l_2$ so that $O_1O_2$ is the common perpendicular to lines $l_1$ and $l_2$ and line $A_1A_2$ forms equal angles with lines $l_1$ and $l_2$. Prove that $O_1A_1 = O_2A_2$.

**Problem 3.6.**
Prove that the line forming pairwise equal angles with three pairwise intersecting lines that lie in plane $H$ is perpendicular to $H$.

**Problem 3.7.**
Points $A_1$ and $A_2$ belong to planes $H_1$ and $H_2$, respectively, and line $l$ is the intersection line of $H_1$ and $H_2$. Prove that the line $A_1A_2$ forms equal angles with planes $H_1$ and $H_2$ if and only if points $A_1$ and $A_2$ are equidistant from line $l$.

## 4 Results on skew lines

**Problem 4.1.**
Consider a plane $H$ and a line $l$ in it. Suppose another line $q$ intersects this plane only at one point. Show $q$ can't be parallel to $l$.

**Problem 4.2.**
Parallel lines $l_1$ and $l_2$ lie in two planes that intersect along line $l$. Prove that $l_1$ is parallel to $l$.

**Problem 4.3.**
Given three pairwise skew lines (i.e. no 2 or them are parallel or intersect). Prove that there exist infinitely many lines each of which intersects all the three of these lines.

**Problem 4.4.**
Given several lines in space so that any two of them intersect. Prove that either all of them lie in one plane or all of them pass through one point.

**Problem 4.5.**
In space, there are given two skew lines $l_1$ and $l_2$ and point $O$ not on any of them. Does there always exist a line passing through $O$ and intersecting both given lines?