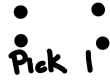


(a) $\binom{4}{1}$



(b) $\binom{4}{3}$



(c) $\binom{5}{2}$



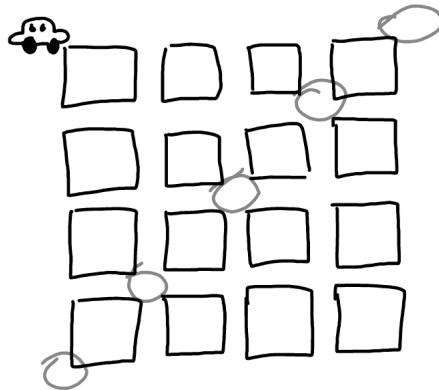
(d) $\binom{5}{3}$



(e) In full sentences, explain how the $\binom{n}{k}$ is related to the taxicab problem triangle.

Problem 3. What is the total number of ways to all the intersections that are 4 blocks away. (Hint: at every intersection, the taxicab can make a choice of going right

or down)



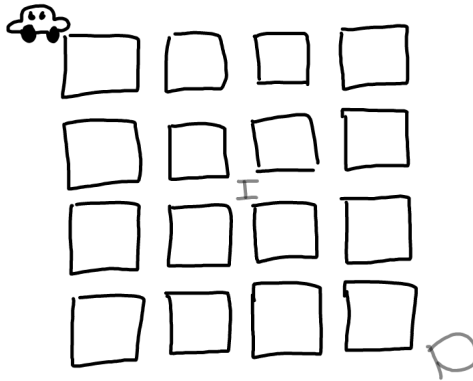
Problem 4. What is the total number of paths to intersections that are n blocks away, and southeast of the driver's current position? Explain your reasoning in full sentences.

Problem 5 (Summed Choice, First time Around). Using the relationship between taxicabs and choice, explain why

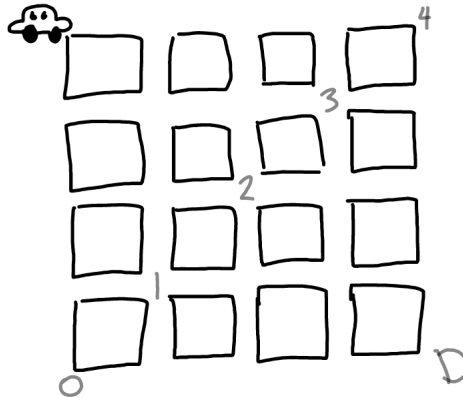
$$\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \binom{n}{3} + \dots + \binom{n}{n-1} + \binom{n}{n} = 2^n$$

Use full sentences.

Problem 6. When Derek goes home, he first wants to stop by Isaac's house to drop off some math circle papers. How many ways can he do that?



Problem 7. Every single path to Derek's house passes through one of his 5 friends' houses.

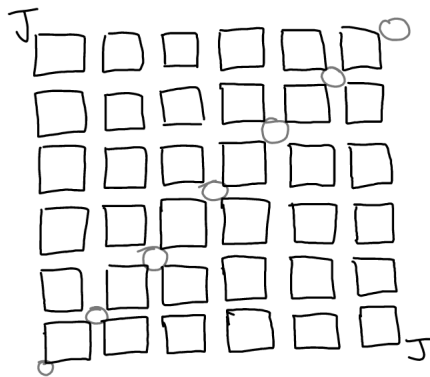


Using this fact, and the previous problem, explain why the number of paths to Derek's house is

$$\binom{4}{0}^2 + \binom{4}{1}^2 + \binom{4}{2}^2 + \binom{4}{3}^2 + \binom{4}{4}^2$$

Explain your answer in full sentences.

Problem 8. Jeff and Jonathan decide to meet up in the middle of the week to discuss math circle problems. Jeff starts at his house, and Jonathan starts at his house, and they decide to stop in the middle. How many ways can they do this? You may leave your solution as an equation. (Hint: Compute the number of ways that they could meet at one of the meeting spots, then add up across all of the meetings spots)



Problem 9 (Bonus). Using the above two problems for inspiration, show that

$$\binom{n}{0}^2 + \binom{n}{1}^2 + \binom{n}{2}^2 + \dots + \binom{n}{n-1}^2 + \binom{n}{n}^2 = \binom{2n}{n}$$

2. REARRANGEMENTS

Another combinatorial thing we may look at is rearrangements. The rearrangement number is the total ways that one can take a few objects and rearrange them. For example, there are 6 ways to rearrange 3 objects

abc acb bac bca cab cba

One way to make sure that you got all of the possible rearrangements is to list them in alphabetical order

Problem 10. List all the arrangements for these four letters in alphabetical order

$abcd$

How many are there?

Problem 11. Here's a method for 5 letters.

(a) First, pick one of the letters. How many ways can you do this?

(b) Then arrange the remaining 4 letters. How many ways can you do this? (Use the previous problem)

(c) Now stick the letter you picked at the start of the arrangement. This gives you a 5 letter arrangement. How many ways can you rearrange 5 letters?

Problem 12. How many ways can you arrange 6 letters? Explain how you got your answer in a full sentence. (You may leave your solution as a formula)

Problem 13. Using the previous problems as inspiration, how many ways can you arrange n letters? Explain your process in full sentences.

For shorthand, we will write

$$n! = 1 \times 2 \times 3 \times \dots n$$

and call this value n *factorial*.

Problem 14 (Choices and Rearrangements). Suppose Morgan wants to select 5 people from the classroom of 25 to help him with a problem. Morgan has the most peculiar method for selecting students

(a) First, Morgan puts all 25 people in a line. How many ways can Morgan do this? (You may leave your solution

(b) Morgan then asks himself,
How many different lines would have had the same 5 people in the front

He concludes that either the first 5 people would have been in a different order, or the last 20 people would have been in a different order. How many different lines have the same first 5 people in the front?

(c) Morgan then notes the following

$$\# \text{ ways to pick 5 from 25} = \frac{\# \text{ ways to arrange 25 people}}{\# \text{ arrangements that have 5 chosen students at front}}$$

How many ways can Morgan choose 5 students from 25? (You may leave this answer as a formula)

Problem 15. Using Morgan's Method, compute these following values

(1) $\binom{4}{2}$

(2) $\binom{5}{2}$

(3) $\binom{6}{2}$

(4) $\binom{6}{3}$

Problem 16. Using full sentences, explain why

$$\binom{n}{k} = \frac{n!}{(k)! \times (n - k)!}$$

Problem 17 ($0!$). What is $\binom{5}{5}$? If we use the above formula to compute $\binom{5}{5}$, what should $0!$ be?

Problem 18 (Summed Choice, Second time Around). (a) What is the number of ways of picking any number of items from n items. (Hint- At every item, decided whether or not you want to pick it.)

(b) What is the number of ways of picking 0 items, or 1 item, or 2 items, or 3 items . . . or n items? (Hint- Sum up the possibilities!)

(c) Why should these two quantities be equal. Set them equal to each other to get a familiar looking identity.

3. SHOPPING FOR CANDY

The grocery store holds 4 types of candy, which are conveniently listed in alphabetical order, and each candy bar costs 1.

- (a) Almond Joys
- (b) Butterfingers
- (c) Candy Canes
- (d) Dumdums

Problem 19. Suppose Isaac goes to the store to buy some candy. He has one dollar. How many ways could he purchase candy?

Problem 20. Suppose Derek goes in with 2 dollars. How many ways could he purchase candy? (Remember that Derek can purchase 2 of the same kind of candy if he wants?)

Problem 21. Morgan, the big spender, enters the store with 3 dollars. How many ways could he purchase candy?

Problem 22. Jonathan decides that he wants to build a candy shopping robot. He knows that the candy in the store is lined up in rows along the aisle, and so programs a robot that can do one of two things:

- Pick up a candy Bar (P)
- Move along the aisle to the next candy bar row (M)

Every program finishes with the robot next to the Dumdum row of candies. The candy is arranged in alphabetical order.

Suppose Jonathan programmed the robot to pick up 2 candy bars. Such a program may look like this

$$PMMPM$$

Which would pick up 2 an Almond Joy, advance 2 rows, and then pick up a Candy Cane, then advance 1 row to the Dumdum aisle and stop.

(a) Describe what the program

$$MPPMM$$

picks up

(b) Write a program that picks up 2 Dumdums

(c) If the Robot is to pick up 2 items, how long does the program have to be?

(d) How many programs can be written that pick up 2 items?

Problem 23. Using Jonathan's Robot, figure out the number of ways to pick out 4 items from the grocery store

Problem 24 (Choose versus Choice with repetition). (a) Suppose Jonathan visits a grocery store with n different types of items, and wants to purchase k items, and only wants to buy at most 1 of each type of item. Then, using the choose notation, how many different kinds of purchases can he make?

(b) Suppose Johnathan is willing to buy more than one of each type of item. Then how many purchases can he make? (Hint: Count the number of programs that there are for a robot to purchase k items at a grocery store with n different types of candy. How long does the program have to be?)

Problem 25 (Summed Choice, Third time Around). Suppose that I have a classroom with n students, and I need to select k of them. One way to do this selection is to decide whether or not to pick the tallest student in the class, and then pick the other students.

(a) If I select the tallest student in the class, then I only need to select $k - 1$ more students, and then I can pick the remaining $k - 1$ students. How many ways are there to do this (You may use choice notation)

(b) If I do not select the tallest student in the class, then I need to select k students. How many ways are there to do this?

(c) With the above problems in mind, fill in the blanks with the appropriate values:

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$$

(d) Using the above identity, show

$$\binom{n}{n} + \binom{n}{n-1} + \dots + \binom{n}{0} = 2\binom{n-1}{n-1} + 2\binom{n-1}{n-2} + \dots + 2\binom{n-1}{0}$$

(e) How does this show us that

$$\binom{n}{n} + \binom{n}{n-1} + \dots + \binom{n}{0} = 2^n$$