

Probability Part II - Repeated Independent Trials

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1 Independent Random Variables

This is a direct continuation of last week's worksheet, so all relevant definitions and examples should be found there.

In this worksheet, we'll continue to study variances. When trying to measure the expected value of a random variable, it makes intuitive sense to measure the random variable again and again, and take the average values. For this approach to work, the variance in our measurements should be low. Before we show this, however, we'll have to introduce some necessary definitions.

Definition 1 Let (Ω, \mathbb{P}) be a finite probability space. Two events A and B are *independent* if

$$\mathbb{P}(A \cap B) = \mathbb{P}(A)\mathbb{P}(B)$$

(that is, if the probability of A and B equals the product of the probabilities of A and B)

Intuitively, two events are independent precisely when one of them doesn't affect the other one. While this doesn't always match the mathematical calculation (so it is no substitute for it), it is good to keep in mind.

Problem 1 Consider an experiment where we roll a fair six-sided dice once. For each pair of events below, decide intuitively, and then mathematically, whether they are independent.

- $A = \{\text{odd number}\}, B = \{\text{number divisible by 3}\}$

- $A = \{6\}, B = \{\text{not } 6\}$

- $A = \{1, 4, 6\}, B = \{2, 6\}$

- $A = \{\text{even number}\}, B = \{\text{prime number}\}$

- $A = \emptyset, B$ is any event

Recall that a random variable is a function $X : \Omega \rightarrow \mathbb{R}$. Since each random variable has corresponding events $\{X = x\}$, we can determine whether those events are independent.

Definition 2 Let (Ω, \mathbb{P}) be a finite probability space. Two random variables X and Y on it are **independent** if the events $\{X = x\}$ and $\{Y = y\}$ are independent for all $x, y \in \mathbb{R}$.

Problem 2 Consider an experiment where we flip one fair coin. Let X be the number of heads, and Y the number of tails. Determine whether X and Y are independent, and show your answer is correct.

Problem 3 Consider an experiment where we flip two fair coins. Let X be the number of heads on the first flip, and Y the number of heads on the second flip. Determine whether X and Y are independent, and show your answer is correct.

Problem 4 Consider an experiment where we flip two fair coins. Let X be the number of heads on the first flip, and Y always equal 1. Determine whether X and Y are independent, and show your answer is correct.

Problem 5 Consider an experiment where we roll a fair die twice. Let X_1 be the value of the first roll, and X_2 be the value of the second roll. Explain (intuitively) why X_1 and X_2 are independent. (Bonus) Prove it.

Problem 6 In general, it seems that the first and second (and third, etc.) trials of an experiment are independent. But can you think of any real-life examples where this might not be the case?

2 Repeated Trials and Sample Mean

From now on, we'll assume successive trials of the same experiment to be independent. We'll also need a stronger property for our random variables.

Definition 3 Let (Ω, \mathbb{P}) be a finite probability space. Two random variables X_1 and X_2 on it are **identically distributed** if they have the same PMF; that is, if $\mathbb{P}(\{X_1 = x\}) = \mathbb{P}(\{X_2 = x\})$ for all real numbers x .

If X_1, \dots, X_n are independent and identically distributed, we will say they are *iid* for short. We now give some examples that suggest that repeated trials of the same experiment will not only be independent, but also identically distributed (so we'll assume this too).

Problem 7 Consider an experiment where we roll a fair die n times, and let X_1 be the value of the first roll, X_2 the value of the second roll, and so on. Show that X_1, \dots, X_n are *iid*.

Problem 8 Consider the same setup as above but with an unfair die. Assuming we are rolling the same unfair die each time, show that X_1, \dots, X_n are still *iid*.

Last week, we estimated the expected value of a fair six-sided die roll by rolling a fair six-sided die repeatedly. What we actually measured was the average, or *mean* of our results. We'll now show that this mean is a good way to measure the expected value.

Definition 4 Let X_1, \dots, X_n be *iid* random variables. The **sample mean** \bar{X} is the random variable

$$\bar{X} := \frac{X_1 + \dots + X_n}{n}$$

Problem 9 Using a computer dice roller (preferably), roll 200 fair six-sided dice, in groups of 10. Write down the 20 sample means (one from each group of 10).

Problem 10 Based on this data, what do you think is the expected value of the sample mean?

Let's prove (or disprove) this conjecture.

Problem 11 Let X_1, \dots, X_n be iid random variables. Show that $\mathbb{E}[\bar{X}] = \mathbb{E}[X_1] = \dots = \mathbb{E}[X_n]$ (Hint: Use Problem 12 from last week.)

3 Variance of the Sample Mean

In order to use the sample mean as a good measurement of the expected value, its variance (which measures how far off it will typically be) should be close to zero. As we did last time, we will do another experiment to show the variance.

Problem 12 Using a computer dice roller, roll 400 fair six-sided dice, in groups of 20. Write down the 20 sample means (one from each group of 20). If you want, you can also repeat this in groups of 30 as well.

Problem 13 Based on this data (and the data from Problem 9), hypothesize a relationship between the number of trials n and the variance of the sample mean. (Hint: It may help to draw some sort of graph.)

In order to prove (or disprove) your conjecture, we'll need some useful facts about variance.

Problem 14 (a) Show that for any (real) constant c , $\text{Var}(cX) = c^2\text{Var}(X)$

(b) (Bonus) Show that if X and Y are independent, then $\text{Var}(X + Y) = \text{Var}(X) + \text{Var}(Y)$.

Problem 15 Let X_1, \dots, X_n be iid random variables. Write down a formula for $\text{Var}(\bar{X})$, in terms of $\text{Var}(X_1)$ and the number of trials n .

Problem 16 (Law of Large Numbers) Let X_1, \dots, X_n be iid random variables. Show that as n becomes larger, $\text{Var}(\bar{X})$ becomes smaller and approaches zero. Hint: try proving first that $\mathbb{E}[XY] = 0$ when X, Y are independent.

Problem 17 (Pythagorean Theorem) Show for independent random variables X, Y that:

$$\mathbb{E}[(X + Y)^2] = \mathbb{E}[X^2] + \mathbb{E}[Y^2]$$

Problem 18 (Wald's Identity, special case) Let X_1, X_2, \dots, X_{20} be iid but necessarily fair 6 sided dice rolls, suppose we roll an additional 20 sided fair dice and call the result of this random variable Y . Suppose in addition that you know that the expectation of each of the X_i s to be 3, calculate the expected value of the sum of the first Y X_i s, namely:

$$\mathbb{E} \left[\sum_{k=1}^Y X_k \right]$$

Problem 19 (Wald's Identity) Generalize the last problem, suppose that Y is a random variable that gives a number between 0 and some large number N , and X_1, \dots, X_N are iid random variables, calculate the value of:

$$\mathbb{E} \left[\sum_{k=1}^Y X_k \right]$$

Problem 20 (FKG inequality, very weak case) Let X be a random variable that takes values in $0, 1, \dots, N$ for some large integer N , suppose that $f, g : \mathbb{N} \rightarrow \mathbb{N}$ be two functions with the property that $f(a) > f(b)$ and $g(a) > g(b)$ whenever $a > b$ are natural numbers. Prove that the random variables $f(X)$ and $g(X)$ satisfy the following:

$$\mathbb{E}[f(X)g(X)] \leq \mathbb{E}[f(X)]\mathbb{E}[g(X)]$$