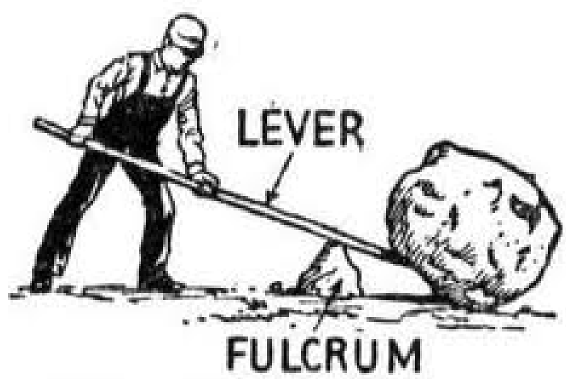
Dr. Oleg Gleizer
prof1140g@math.ucla.edu

Geometry of Masses, Lesson 1

A *lever* consists of a solid beam rotating around a fixed point, a *fulcrum* or *pivot*. The force applied to one side of the lever results in the force being exerted at the opposite side.

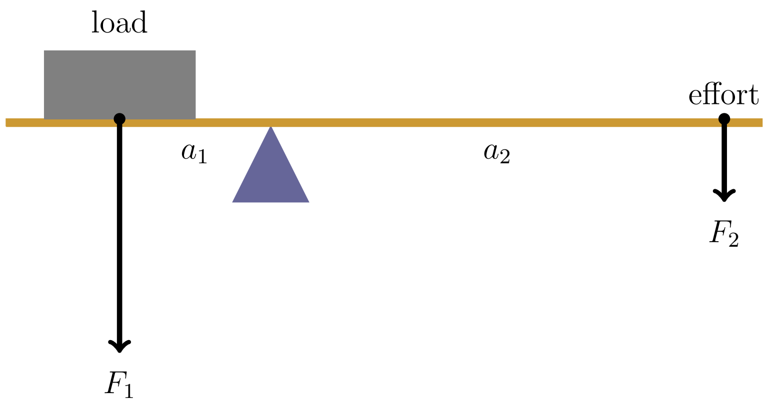


Man lifting a stone with a lever

The distance a from the fulcrum to the point where the force F is applied is called the *arm* of the force. The product

$$T = aF \tag{1}$$

is called the *moment of force* or *torque*. The arm on the load side, a_1 on the picture at the top of the next page, is called the *resistance arm*, the arm on the opposite side (a_2 on the picture below) is called the *effort arm*.



The lever is in balance when the torque of the load equals to that of the effort.

$$a_1 F_1 = a_2 F_2 \tag{2}$$

The load starts moving when the torque of the effort exceeds the torque of the load.

Problem 1 *The curb weight*[1] *of a Toyota Sienna minivan is 4,000 lbs. A person weighing 200 lbs wants to lift the car with a lever. The resistance arm of the lever is 3'. How long should be the effort arm?*

Answer: *The efffort arm should be* __________*' long.*

[1]The total weight of a vehicle with standard equipment and all the necessary liquids, including a full tank of fuel, motor oil, coolant, etc. while not loaded with either passengers or cargo.

Problem 2 *A boy weighing 50 lbs wants to lift his 200 lbs father using a 5-foot-long stick as a lever. Where should he place the fulcrum? Hint: a picture will help.*

Answer: *The boy should place the fulcrum at* __________________________.

Let x be a point on the number line in between zero and one.

$$0 < x < 1$$

Consider the segment $[0, 1]$ as a lever. The pair of weights $(w_0 : w_1)$ a person should place at zero and one respectively so that the lever with the fulcrum at x would be in balance is called the *barycentric coordinates* of x.

Problem 3

a. *Let* $x = 0.25$ *and* $w_0 = 3$. *Find* w_1.

b. *Find* w_1 *for the same* x *and* $w_0 = 12$.

c. *Find* w_1 *for the same* x *and* $w_0 = 30$.

d. *Find* w_1 *for the same* x *and* $w_0 = 60$.

e. *Does the ratio* $w_0 : w_1$ *in parts a – d change? Why or why not?*

Problem 3 shows that barycentric coordinates $(w_0 : w_1)$ of a point are unique up to a common non-zero factor. Coordinates of this kind are called *projective coordinates.* Barycentric coordinates are a special, and very important, type of projective coordinates.

Problem 4 *Find barycentric coordinates of the following points. Draw pictures if needed.*

- 1/2

- 1/7

- 0.8

The problem continues to the next page.

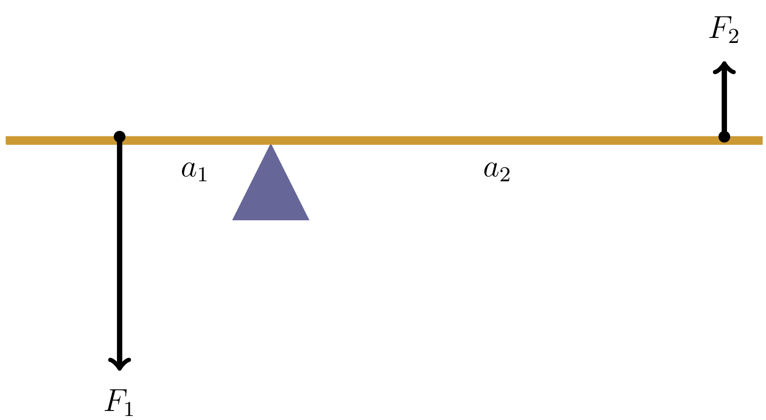
- 0.99

- 0

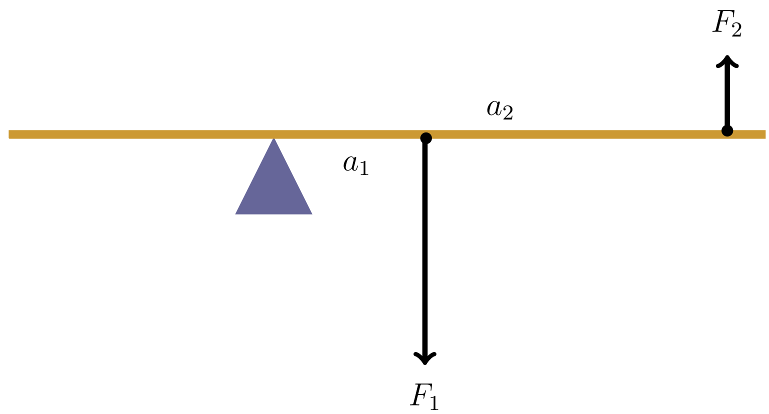
- 1

Question 1 *Can we extend barycentric coordinates to the entire number line? How?*

The idea is to use negative weights. If a positive weight pushes the lever down, then a negative weight pushes it up! However, if we place two weights, one positive and one negative, at the opposite sides of the fulcrum, their torques will rotate the rod in the same direction. We will get not a see-saw, but a merry-go-round!

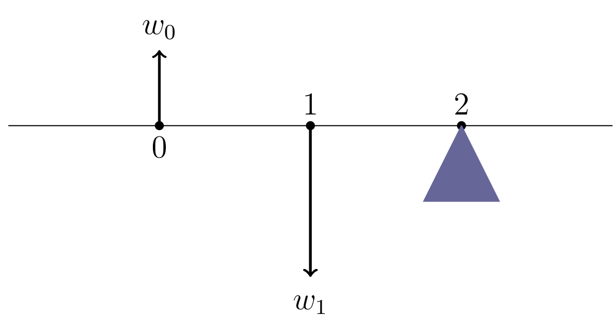


Placing two weights of opposite signs at the same side of the fulcrum brings the lever back to the normal mode of operation.



Example 1 *Find barycentric coordinates of the point 2.*

Since the points 0 and 1 are on one side of the lever, we must use weights of opposite signs.



To have the lever with the fulcrum at 2 in balance, we need the weights to satisfy the following equation.

$$2w_0 + w_1 = 0$$

The weights $w_0 = -1$ *and* $w_1 = 2$ *satisfy the formula above, so barycentric coordinates of the point 2 are* $(-1 : 2)$.

Note that any pair of real numbers with the same ratio will do. For example, $(3 : -6)$ *are also barycentric coordinates of the same point.*

Problem 5 *Find barycentric coordinates of the following points. Draw pictures if needed.*

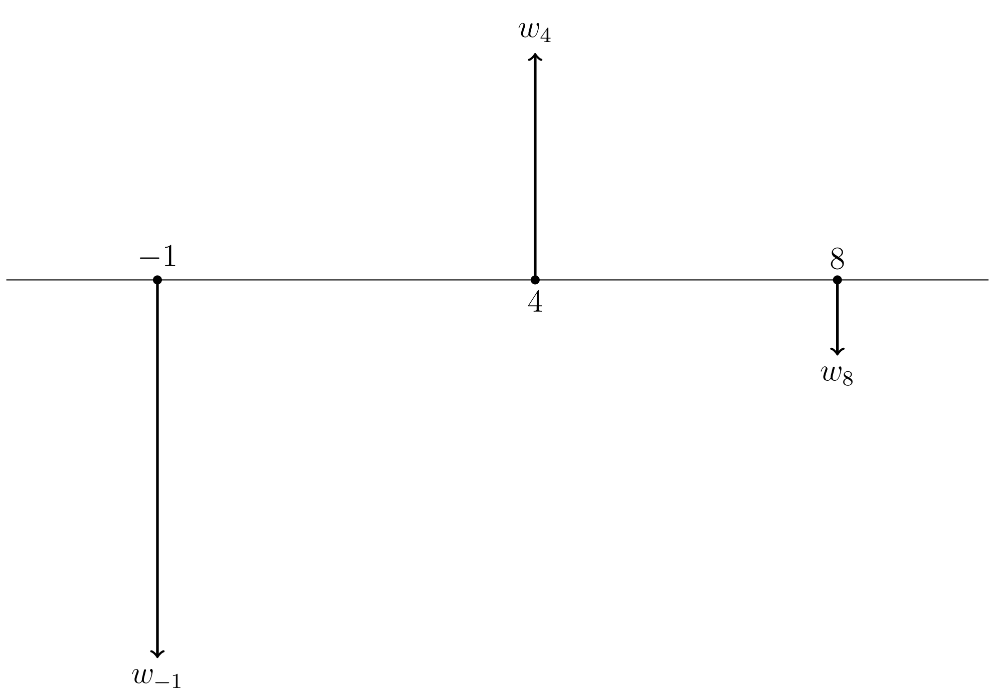
- 3
- 4
- 100
- -5
- -12
- 2.25

We have chosen the points zero and one as the reference points. However, this choice is not essential, any two different points on the number line will do.

Problem 6 *What weights, w_{-5} and w_7 should we place at the points -5 and 7 to have a lever with the fulcrum at -4 in balance?*

Note that we can have more than two forces acting on a lever. Just as above, the lever will be in balance when the sum of the torques on one side of the fulcrum equals to the sum of torques on the other.

Example 2 *Where should we place the fulcrum to have the lever with the weights $w_{-1} = 5$, $w_4 = -3$ and $w_8 = 1$ (located at the points -1, 4, and 8 respectively) in balance?*



If the fulcrum is placed at x, then the arm lengths of the forces w_{-1}, w_4 and w_8 are $x-(-1)$, $x-4$ and $x-8$ respectively. The lever will be in balance if the torques satisfy the following equation.

$$5(x+1) - 3(x-4) + 1(x-8) = 0$$

Solving the equation gives

$$x = -3.$$

The point -3 is called the *center of mass* of the above system. If we hang a weightless straight line equipped with the above weights on a thread at the point, it will not rotate. If hanged at any other point, it will rotate due to the non-zero net torque.

Problem 7

- *Find the center of mass of the weights w_4 and w_8 from Example 2.*

- *Move both weights w_4 and w_8 to their center of mass. Find the center of mass of the weight $w_4 + w_8$ placed at the point and of the weight w_{-1}.*

- *Find the center of mass of the weights w_{-1} and w_8 from Example 2.*

The problem continues to the next page.

• *Move both weights* w_{-1} *and* w_8 *to their center of mass. Find the center of mass of the weight* $w_{-1} + w_8$ *placed at the point and of the weight* w_4*.*

• *Compare the answers for the second and fourth part of this problem to that of Example 2. Do you always get the same point? Why?*

Problem 7 is an example of the below small piece of theory at work.

To figure out the center of mass of a system of point-weights, one can take the following steps.

1. Choose a subsystem of the system and find its center of mass.

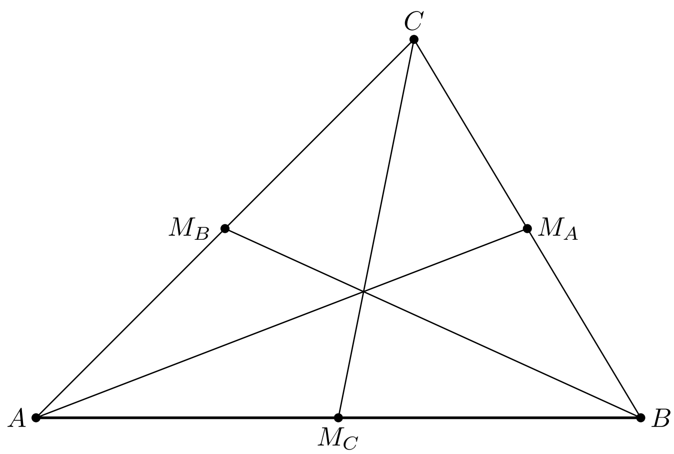
2. Move all the weights of the subsystem to the center of mass of the subsystem, add them up, and consider as one single weight.

3. Find the center of mass of the new system.

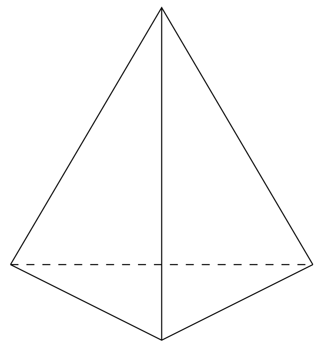
Theorem 1 *The final result is independent of the choice of the subsystem.*

We will prove theorem 1 later.

Problem 8 *Show that medians of a triangle intersect at one point and that the point divides each median in the ratio* 2 : 1 *counting from the opposite vertex.*



A *tetrahedron*, also known as a *triangular pyramid*, is the simplest possible solid in the Euclidean 3D.



Each face of a tetrahedron is a triangle. Imagine that we find the intersection point of the medians for every face of a tetrahedron and connect the point to the opposite vertex.

Problem 9 *Show that all the four resulting lines intersect at one point. Further show that the intersection point divides each of the lines in the ratio 3:1 counting from the corresponding vertex.*

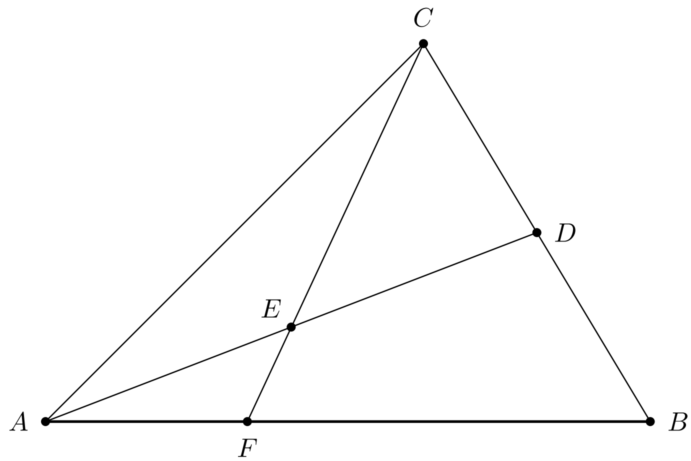
Let us call the edges of a tetrahedron *opposite*, if they do not have common points.

Problem 10 *Show that the lines connecting centerpoints of the opposite edges of a tetrahedron all meet at one point. Further show that it is the same point as in problem 9.*

Problem 11 *Draw a 4-dimensional triangular pyramid, also known as a pentachoron.*

Problem 12 *Generalize the statements of problems 8 and 9 to four dimensions. Prove the statement.*

Problem 13 *D is a centerpoint of the side BC in a triangle ABC. E is the centerpoint of the median AD. F is the intersection point of the lines AB and CE. Find the the following ratios.*



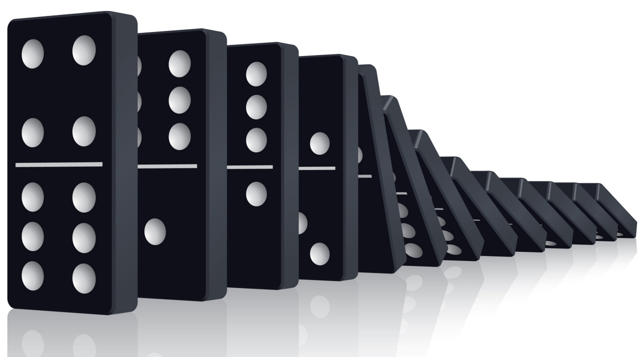
- $CE \div EF =$ ____________________

- $AF \div FB =$ ____________________

To prove theorem 1 by induction on the number of point-weights in the system, let us have a short review of mathematical induction first.

Suppose that we have an infinite list of related mathematical statements S_n where n are natural numbers. The first statement is called the *base case*. Suppose that S_1 is true. If we establish the *inductive step* by showing that S_n implies S_{n+1}, then we prove the validity of the statements S_n for any and all $n \in \mathbb{N}$. Indeed, $S_1 \Rightarrow S_2$, $S_2 \Rightarrow S_3$, $S_3 \Rightarrow S_4$, and so forth.

An example of mathematical induction is the *domino effect*. Imagine that we have an infinite set of dominoes lined up at equal distances along a straight line. Imagine further that the distance between the dominoes is short enough for a falling domino to force the fall of the next one.



Let us prove an infinite list of related statements

$$S_n = the\ nth\ domino\ falls$$

by induction.

The base case: the first domino falls. We prove it by inspection. Give the first domino a nudge and see what happens. If it falls, this proves the base case. If it doesn't, then the domino effect may not occur.

The inductive hypothesis: assume that S_n is true, the nth domino falls.

The inductive step: thinking S_n is true, prove that S_{n+1} is true as well. Proof – the falling nth domino forces the fall of the $n + 1$ one.

This way, the fall of the first domino forces the fall of the second, the fall of the second forces the fall of the third, and so forth.

The following famous formula

$$\sum_{i=1}^{n} i = 1 + 2 + 3 + \ldots + n = \frac{n(n+1)}{2} \tag{3}$$

was anecdotally discovered by Gauss at the age of three. (Outside of mathematical texts, the Greek letter Σ is pronounced as *sigma*. In mathematical texts, it means and reads a *sum*.) Here is Gauss's proof. Let us write down the sum twice, reversing the order of the summands the second time.

$$\begin{array}{rcccccccccccc} \Sigma = & 1 & + & 2 & + & \ldots & + & (n-2) & + & (n-1) & + & n \\ \Sigma = & n & + & (n-1) & + & \ldots & + & 3 & + & 2 & + & 1 \end{array}$$

Adding the sums term-by-term produces the following.

$$2\Sigma = (n+1) + (n+1) + \ldots + (n+1) + (n+1) = n(n+1)$$

Dividing both sides by two proves (3).

Problem 14 *Use mathematical induction to prove formula (3).*

Problem 15 *Use mathematical induction to prove the following.*

$$\sum_{i=1}^{n} i^2 = 1^2 + 2^2 + 3^2 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6} \tag{4}$$

Problem 16 *Use mathematical induction to prove theorem 1. Hint: you have established the base solving problem 8.*