GRAPHS II

MATH CIRCLE (INTERMEDIATE) 1/20/2013

A graph is called **connected** if any two of its ____________________ can be connected by a ____________________.

A path is a sequence of ______________________, each of which begins at the endpoint of the previous one.

A _________________ (or ______________________) is a closed path: a path in which the starting and ending vertices coincide.

A _____________________________________ is a part of a graph that forms a connected graph on its own, but is not necessarily connected to the rest of the graph.

(1) In the country of Seven there are 15 towns, each of which is connected to at least 7 others. Prove that one can travel from any town to any other town, possibly passing through some towns in between.
(Hint: Prove it by contradiction. Pick two towns at random, and suppose there is no path connecting them. If each town is still connected to at least 7 others, how many towns must there be?)

(2) Prove that a graph with n vertices, each of which has degree $\frac{n-1}{2}$, is connected.

(3) In Never-Never-Land there is only one means of transportation: magic carpets! 21 carpet lines serve the capital. A single line flies to Farville, and every other city is served by exactly 20 carpet lines.

(a) State the theorem you learned last week regarding the numbers of odd vertices you can find in a graph.

(b) Use that theorem to prove that it is possible to travel by magic carpet from the capital to Farville (perhaps transferring from one carpet line to another). (Hint: Prove it by contradiction.)

(4) In a certain country, 100 roads lead out of each city, and one can travel along those roads from any city to any other. One road is closed for repairs. Prove that one can still get any city to any other.

(5) **Discovering Eulerian paths.** In the following problems, when you are asked to draw a graph, if it is not possible, say so and explain why.

(a) Draw a graph with six vertices, one of which is odd.
(note: the fact that the graph has six vertices is arbitrary.)

(b) Can you fully re-draw the graph you drew in part (a) without lifting your pencil from the paper? If so, circle the starting and ending points.

(c) Draw a graph with six vertices, two of which are odd.

(d) Can you fully re-draw the graph you drew in part (c) without lifting your pencil from the paper? If so, circle the starting and ending points.

(e) Draw a graph with six vertices, three of which are odd.

(f) Can you fully re-draw the graph you drew in part (e) without lifting your pencil from the paper? If so, circle the starting and ending points.

(g) Draw a graph with six vertices, four of which are odd.

(h) Can you fully re-draw the graph you drew in part (g) without lifting your pencil from the paper? If so, circle the starting and ending points.

(i) Given the following information:

- The pattern you found in parts (a)-(h) is *independent* of the number of vertices in the graph.
- You can draw a graph with zero odd vertices without lifting your pencil from the paper.
- It is *impossible* to fully draw any graph with more than five odd vertices without lifting your pencil from the paper.
  (You can verify these statements at home, but we'll spare you the dirty work here.)

  Formulate a hypothesis that summarizes your results from parts (a)-(h) and agrees with the given information.

IMPORTANT INFORMATION

This sort of graph was first studied by the great mathematician Euler, in connection with a classic problem about the Konigsburg bridges (which we will have a look at later). An **Eulerian path** is a path that accomplishes this. An **Eulerian cycle (Eulerian circuit)** is an Eulerian path in which the starting point and endpoint coincide.

(6) Proving by contrapositive that:
~ If a graph has an Euler cycle, then every one of its vertices has even degree. ~

(a) What is the contrapositive of the statement
"If a graph has an Euler cycle, then the degree of every vertex is even."?
(Hint: Remember, the contrapositive of "if A, then B" is "if not A, then not B", and a statement and its contrapositive *are logically equivalent*)

(b) If a vertex has odd degree, why can't every one of its edges belong to a cycle?
(Hint: A cycle "enters" and "leaves" a vertex along its edges an equal number of times).

(c) Why does this show that if the graph has at least one vertex with odd degree, there are no Euler cycles? How does this prove our initial conjecture (in bold above)?

(7) Can you explain why a graph that has exactly *two* odd vertices can have an Euler path, despite not having an Euler cycle?
(Hint: Look at your answers to Problem 6. Specifically, which vertices are circled...)

(8) A map of the city of Konigsburg is given in Figure 1. The city lies on both banks of a river, and there are two islands in the river. There are seven bridges connecting the various parts of the city. Can one stroll around the town, crossing each bridge exactly once? Why or why not?

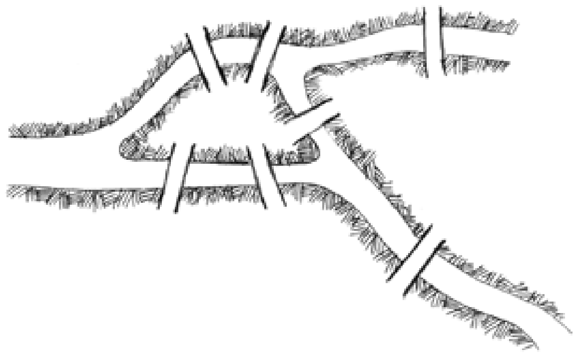

FIGURE 0.1. The bridges of Konigsburg.

(9) The wire test.
   (a) A piece of wire is 120 cm long. Can one use it to form the edges of a cube, each of whose edges is 10 cm, without cutting the wire?

   (b) What is the smallest number of cuts one must make in the wire, so as to be able to form the required cube?

(10) Sony has an obsession with sorting things. He is having a party next week, and decides that he will rank the 50 guests by popularity. He will say that one person is more popular they have more friends at the party.

(a) Can you make this problem into a graph? What are the vertices? The edges?

(b) (Review problem - pigeonhole principle) Prove that there will be two people at the party with the same popularity (number of friends).

(11) Can you write the digits $1-9$ in a row so that the sum of adjacent values are divisible by 5, 7 or 13? For example, $4-9-1$ works, but $1-6-2$ does not. (Hint: Turn the problem into a graph! What should the vertices be? How about the edges?)

- Math Kangaroo Problems -

(12) What is the ratio of the area of a regular hexagon with side length 1 to the area of an equilateral triangle with side length 3?

(13) Water from Faucet A fills the swimming pool in 10 hours. Water from each of the other two faucets (B and C) fills the same pool twice as fast. In how many hours will the swimming pool be filled if all three faucets are open?

(14) Quadrilateral $ABCD$ is a square (see Figure 2). The measure of angle OND is 60 degrees. What is the measure of angle COM?

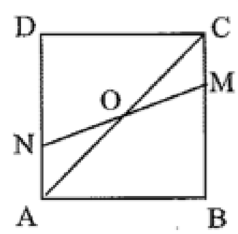

FIGURE 0.2. Problem 14.

(15) The product of my children's ages is 1664. The oldest child is twice the age of the youngest. How many children do I have, and what are their ages?

(16) What is the first digit of the smallest natural number in which the sum of the digits equals 2011?