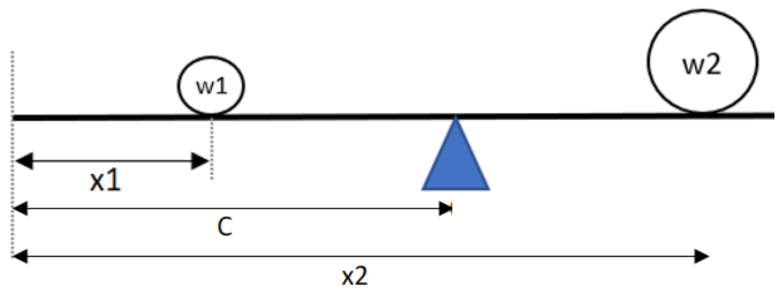
Probability, Monte Carlo Simulation & Application in Finance – Day 3 (Return)

Warm-up

Problem (a) Moment: In physics, the moment of force acting on an object, is the product of the force and the perpendicular distance from a reference supporting point. In the below figure, two weights w1 and w2 are placed on a rod. Weight w1 is at length x1 from the left end of rod while weight w2 is at length x2 from the left end of rod.



i) Find the length C from the left end of the rod at which the two weights will balance each other.

ii) If the proportion of total weights p1=w1/(w1+w2) and p2=w2/(w1+w2), rewrite C in terms of p1, p2, x1 and x2.

The expected value of a random variable X with possible outcomes (values) x_1, x_2,……x_k with respective probabilities p_1, p_2,……p_k is the probability weighted sum of each possible value. It is also referred by few other ways, such as the mean value (mu), E[X], X bar, the first moment.

$$\mu \equiv \mathrm{E}[X] = \overline{X} = \langle X \rangle = x_1 * p_1 + x_2 * p_2 + \dots + x_k * p_k$$

The *moments* of a random variable X are the expected value of the powers of the random variable itself.

$$\mu_\ell \equiv \mathrm{E}\left[X^\ell\right] \equiv \langle X^\ell \rangle = x_1^\ell * p_1 + x_2^\ell * p_2 \dots\dots + x_k^\ell * p_k$$

So, the second moment of random variable X will be equal to: $x_1^2 * p_1 + x_2^2 * p_2 \dots\dots + x_k^2 * p_k$

Problem (b): Let X be a variable that always takes a value of m. Find:

1. The first moment of X aka its expected value.

2. The second moment of X.

Problem (c): Suppose there is an amoeba. Every minute, this amoeba can either die, do nothing, split into 2, or split into 3 amoebas; all these scenarios being equally likely to happen. All further amoebas behave the same way:

1. What is the expected number of off-spring amoebas?

2. What is the second moment of the number of off-spring amoebas?

3. What is the probability that the entire population of amoeba will eventually die off?

Asset, Prices & Returns

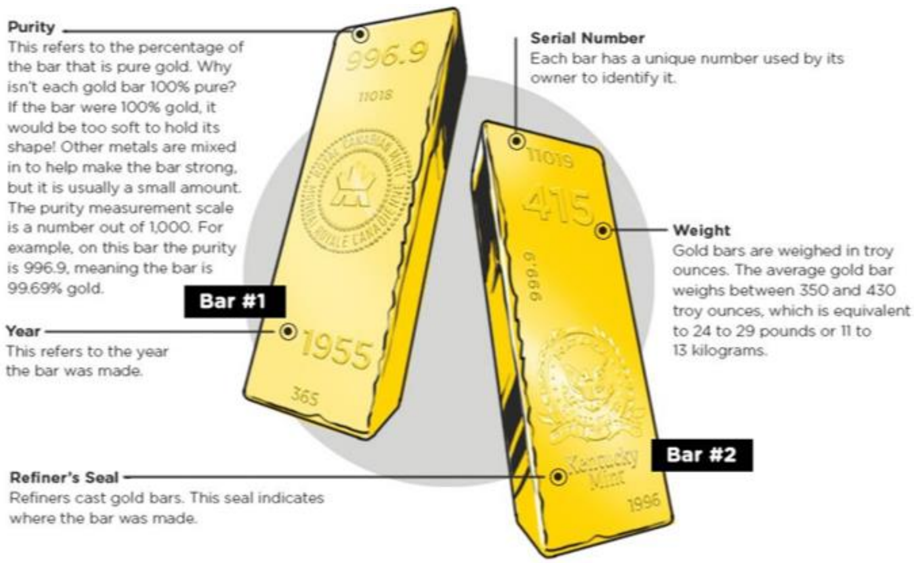
An “asset” is like a special treasure that you own. It could be a toy, a bike, a Pokémon trading card or even money in your piggy bank. These treasures are valuable because they can help you get other things you need or want later. You can use it now or trade it in future. For example, if you have a savings account, the money in it is an asset because you can use it to buy things you need now or want later. It’s something you own that has a worth, and it can be anything from money and property to skills and knowledge.

The worth or value or price of an asset depends on:

- how hard or easily its available (i.e., its supply, e.g., a scarce mineral resource like gold or diamond will be worth more than minerals who are more easily available like iron or copper)
- how much of utility it provides (i.e., its demand, e.g., Lithium demand is increasing due to rise of electric vehicle manufacturing)

Problem (d): Most countries have a central bank that manages the nation's currency and money supply. Examples include the Federal Reserve in the United States, the European Central Bank in the Eurozone, and the Bank of Japan. In many countries, the central bank is responsible for printing money and minting coins. In some cases, other government agencies or institutions may handle the physical production of currency. Why can't the central bank or such agencies print and issue unlimited amount of money for their respective country and make their respective country ‘rich’?

Problem (e): You can collect valuable items that might get even more valuable over time, like rare Pokémon cards or comics. Investing is when you buy something with the expectation that it will earn you money over time. One of such items people “invest” in is Gold. Closely observe the details of each gold bar below and determine how much these bars are worth today if today’s gold price is $2700 per ounce.



Problem (f): People have always thought gold is special because it's shiny, doesn't rust, and is pretty rare. They buy gold and keep them because they believe gold will be worth more in the future. When the value of gold goes up, they can sell it for more money than they paid for it originally. Assuming there is no additional earning from holding the gold nor there is any cost to hold/store the bought gold, the return on investment can be calculated in % as:

$$\text{Return} = ((\text{Final Value} - \text{Initial Value}) \,/\, \text{Initial Value}) \times 100\%$$

You buy gold for $1,200 per ounce and hold onto it for 1 year. Over this period, the value of gold appreciates to $1,500 per ounce. There are no additional earnings from the gold during the holding period nor there is any cost to hold/store the bought gold. Calculate the return of this gold investment.

Problem (g): Below table shows the average gold prices for last 10 year. Find the return for each year

Year	Average Gold Price (USD per ounce)	Return (%)
2024	$2,662.28	
2023	$2,045.00	
2022	$1,800.00	
2021	$1,770.00	
2020	$1,770.00	
2019	$1,392.00	
2018	$1,268.00	
2017	$1,257.00	
2016	$1,250.00	
2015	$1,160.00	

Problem (h): Find average Return for 10-year period. Let's call it 'mu' µ. What will be its unit?

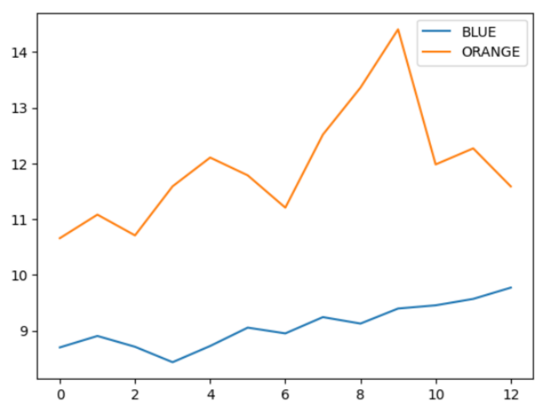
More practice from price to returns

Problem (i): Read the monthly prices (numbers are in $ million) of 2 assets "BLUE" & "ORANGE" house and plot them as line chart. Since typing is tedious, let's read the data from a file. Pandas provides a convenient and simple way to read in a CSV file of the returns.

```
import pandas as pd
prices = pd.read_csv('https://github.com/azayverma/data/blob/main/sample_prices.csv?raw=true')
prices
```

BLUE	ORANGE
8.7000	10.6600
8.9055	11.0828
8.7113	10.7100
8.4346	11.5907
8.7254	12.1070
9.0551	11.7876
8.9514	11.2078
9.2439	12.5192
9.1276	13.3624
9.3976	14.4080
9.4554	11.9837
9.5704	12.2718
9.7728	11.5892

```
prices.plot()
```



Problem (j): Compare these two-line charts of house prices and write down how they are different.

Problem (k): Which asset/house you will invest in or buy and why?

Problem (l): Find how much each asset's price increase or decrease each month.

```
prices.pct_change()
```

BLUE	ORANGE
NaN	NaN
0.023621	0.039662
-0.021807	-0.033638
-0.031763	0.082232
0.034477	0.044544
0.037786	-0.026381
-0.011452	-0.049187
0.032676	0.117008
-0.012581	0.067353
0.029581	0.078249
0.006151	-0.168261
0.012162	0.024041
0.021149	-0.055623

The values in above tables are called monthly 'price returns' for respective asset. Note how we cannot compute returns for the first month, because we don't have the closing price for the previous month. In general, we lose one data point when we go from prices to returns.

The return from time t to time ${t+1} is given by:

$$R_{t,t+1} = \frac{P_{t+1} - P_t}{P_t}$$

or alternately

$$R_{t,t+1} = \frac{P_{t+1}}{P_t} - 1$$

Problem (m): Sometimes an asset can give additional income:

- One such asset is “stock” - A stock is actually a piece of a company. It's not a physical piece, like a brick or window, but a part of the ownership of a company. Stocks can be bought and sold through the stock market, and they can have different prices depending on the company and how it's doing at the time. The value of the stock depends on the profits of the company and changes constantly. Companies usually issue stock to raise money for a variety of reasons, including expanding or modernizing their operations. You can buy stocks or portion of companies, say Tesla and the company periodically will share its profit and give its stockholder additional income which is called ‘dividend’.

- Similarly, you can “loan” your money to individuals, banks, companies, government, etc and they will give you periodic income in form of ‘interest’. “Bonds” are issued by governments and corporations when they want to raise money. By buying a bond, you're giving the issuer a loan, and they agree to pay you back the value of the loan on a specific date (called maturity date), and also pay you periodic interest payments along the way, usually twice a year.

In our case here, you can rent out the house you bought to gain rental income every month. Rewrite the return formula above to find your new ‘return’ from time t to t+1 assuming you have a monthly rental income from the house bought.

Problem (n): Suppose an investor buys a bond worth $1000 which gives interest at a rate of 5% on amount invested for a maturity period of 3 years. Assuming annual interest payment, how much will the investor receive in interest payment each year?

Problem (o): Alice invests $1000 in a 3-year bond. The bond pays back annually 5% compound interest rate on the amount invested. How much money will Alice have at the end of 3 years?

Problem (p): Sometimes we need to incur a cost to hold, store or maintain an asset. In our case, we need to spend money on regular maintenance of the house. If we are buying gold, we might prefer to keep it safe in a bank locker for which we will have to pay a fee. Rewrite the return formula from problem m above to find your new 'return' from time t to t+1 assuming you have a monthly maintenance cost for the house bought.

When we account for all the money we paid and received while holding an asset during a period, the calculated return is called 'Total Return (TR)'

$$TR_{t,t+1} = \frac{P_{t+1} + Income_{t,t+1} - HoldingCost_{t,t+1}}{P_t} - 1$$

Almost all the problems in financial analysis looks at total returns instead of only price returns.

Problem (q): Total return of an asset can be greater than or less than or equal to the price return of the asset. Select True/False

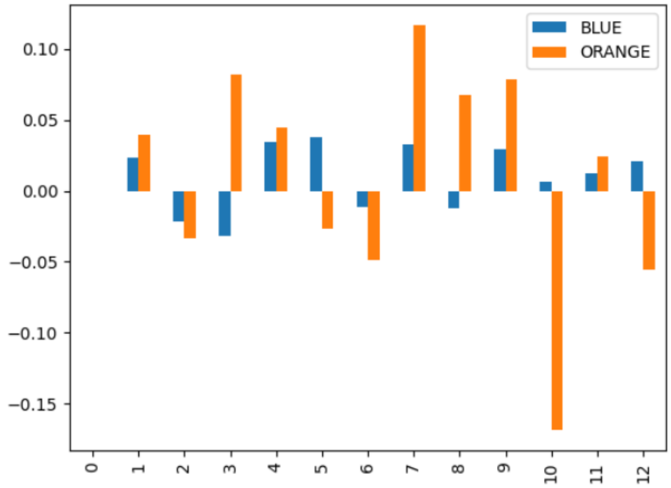
Problem (r): Find mean of monthly returns for both houses.

```
returns = prices.pct_change()
returns.mean()
```

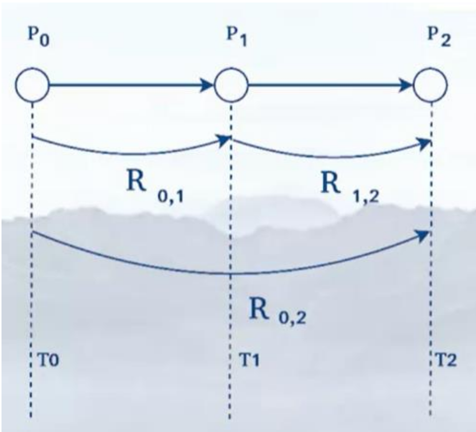
Will your choice of house to buy/invest change now that you know the mean return of each? Why?

Problem (s): Create a bar chart showing monthly returns of 2 assets. As you observed above, the mean of their respective monthly returns is same. Will your choice of house to buy/invest change now that you see how the returns vary for each asset? Why? Which asset return is more volatile (that changes quickly and suddenly giving unstable or riskier situation)? Which asset is riskier?

```
returns.plot.bar()
```



Problem (t) Multi-Period Return: If you know the return from period t to t+1 and the return from period t+1 to t+2, what will be the return from period t to t+2?



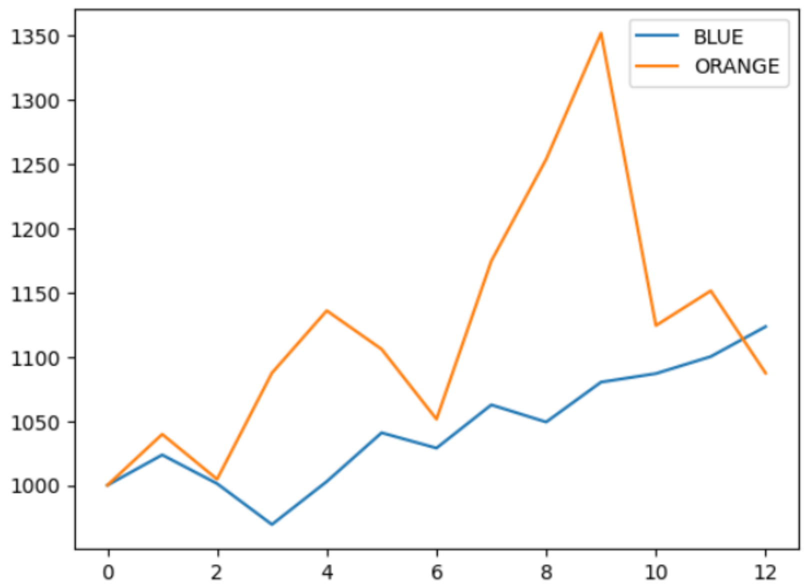
Problem (u): Let's say you buy an asset that returns 10% on the first day, and it loses 3% on the next day. How much % you gained or lost in these 2 days?

Problem (v): Before the holidays, a toy store raised prices by 25%. By what percentage should the store drop the prices after the holidays so that they return to the original prices?

Problem (w): A bookseller sold a book with a 5% discount and got a 10% profit. What would have been their profit without the discount?

Problem (x): You buy a stock at the closing price on Monday. On Tuesday, the stock closes at 10% above Monday's closing price. On Wednesday, it falls and closes at 10% below Tuesday's closing price and you sell it at that closing price. Did you have a negative, positive or zero return?

Problem (y): Invest $1000 in each of the 2 assets (BLUE & ORANGE) in our sample data set above, at the very start of the period and find the amount it will grow at the end. Plot the monthly progress. Which one will end higher?



You can see the blue asset sort of chugs along and does fairly well, and the orange asset is all over the place. And for a good part of the year, it looks like the orange asset might have been the smart thing to be in, but it actually ends the year lower. That's interesting, because the average return of both of these assets is exactly the same.

So, the first thing that you should take away from this is, just because the average monthly return, is the same, doesn't mean that you're going to end up with the same amount of money. In fact, at the end of month 12, you have actually different values.

Annualizing Returns

Problem (1): An asset gives 1% return every month. What will be its annual return? (Hint: use your understanding of multi-period returns above). Will it be less than or more than or equal to 1%*12 times = 12%?

Problem (2): A stock gains 1% over a quarter (i.e., a 3-month period). What is its annualized return?

To annualize a return for a period, you compound the return for as many times as there are periods in a year. For instance, to annualize a monthly return you compound that return 12 times.

To annualize a monthly return R_m you would use:

$$(1 + R_m)^{12} - 1$$

To annualize a quarterly return R_q you would use:

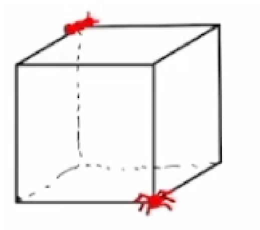
$$(1 + R_q)^{4} - 1$$

And finally, to annualize a daily return R_d you would use:

$$(1 + R_d)^{252} - 1$$

Problem (3): Why 252 has been used to convert daily returns to annual returns?

Homework Problem 1 (The Blind Spider): An ant and a blind spider are on opposite corners of a cube. The ant is stationary and the spider moves at random from one corner to another along the edges only. What is the expected number of turns before the spider reaches the ant?



Homework Problem 2: An investment of 10,000$ grows to 15,000$ in 5 years. What is the annual compound interest rate?