GRAPHS I

MATH CIRCLE (INTERMEDIATE) 1/13/2013

(1) **Counting edges.** For the following problems, assume that the value of n is such that the graph exists.

(a) We have n houses in a village. Every house is connected to exactly one other house by a road. What is the total number of roads in this village?

(b) Suppose $n > 2$. Now suppose every house is connected to exactly two other houses. What is the total number of roads in this village?

(c) Suppose $n > 4$. Now suppose every house is connected to exactly four other houses. What is the total number of roads in this village?

(d) Suppose $n > m$. Now suppose every house is connected to exactly m other houses. What is the total number of roads in this village?

(2) In Smallville there are 15 telephones. Can they be connected by wires so that each telephone is connected with exactly five others? Justify your answer.
(Hint: Use your results from Problem 1.)

(3) In a certain kingdom, there are 100 cities, and four roads lead out of each city. How many roads are there altogether in the kingdom?

(4) **Proving a theorem.**

(a) Can you relate the sum of the degrees of the vertices of a graph to the number of edges the graph contains? (It might help to draw out a few pictures.)

(b) Is the sum of the degrees of the vertices of a graph even or odd? How do you know?

(c) We say that a vertex is **even** if it has even degree.
We say that a vertex is **odd** if it has odd degree.
Theorem: The number of odd vertices in a graph must be even.
Prove this theorem, by using your answer to part (b).
(Hint: Under what condition will the sum of a series of odd numbers be even?)

(5) There are 30 students in a class. Can it happen that 9 of them have 3 friends each (in the class), eleven have 4 friends each, and ten have 5 friends each?

(6) There are 30 students in a class. Can it happen that 20 of them have 19 friends each (in the class), and 10 of them have 9 friends each?

(7) Prove that the number of people who have ever lived on earth, and who have shaken hands an odd number of times in their lives, is even.

(8) Can 9 line segments be drawn in the plane, each of which intersects exactly 3 others? Prove or disprove it!

(9) Is it possible to write all the natural numbers 1 through 100 in a row in such a way that the (positive) difference between any two neighboring numbers is not less than 50?

(10) A chessboard has the form of a cross, obtained from a $4x4$ chessboard by deleting the corner squares. Can a knight travel around this board, pass through each square exactly once, and end on the same square he starts on?

(11) In the country of Figura there are nine cities, with the names 1, 2, 3, 4, 5, 6, 7, 8, 9. A traveler finds that two cities are connected by an airplane route if and only if the two-digit number formed by naming one city, then the other, is divisible by 3. Can the traveler get from City 1 to City 9?

- Review Problems -

(12) Prove that the product of any five consecutive natural numbers is divisible by 30. Your answer **must** contain the words "relatively prime."

(13) In how many ways can I arrange 5 different science books and 6 different history books on my bookshelf, if I require that there are science books on both ends?

- Math Kangaroo Problems -

(14) What is the smallest number of rectangular blocks with sides of 2cm$x$6cm$x$1cm needed to make a cube?

(15) Side AC of triangle ABC is divided into 8 equal parts by 7 segments parallel to side BC. If $|BC| = 10$, can you find the sum of the lengths of these 7 segments?

(16) All the positive whole numbers which are equal to the product of their factors are written in ascending order. What is the sixth number that will be written?

(17) What is the last digit of the number $\frac{1}{5^{2000}}$ in decimal notation?

(18) What is the measure of the angle formed by the hour hand and minute hand of a clock at 4:40PM?