Interpolation

ORMC Olympiads (adapted from Sirius Center)

January 19, 2024

Theorem: The number of distinct roots of a polynomial $A(x)$ is at most $\deg A$.

Exercises

Problem 1. Are the following functions polynomials?

a) $\sin x$;

b) $\frac{1}{1+x^2}$.

Problem 2. Let the values of polynomials $F(x)$ and $G(x)$ coincide at n distinct values of the variable, and the degrees of these polynomials are less than n. Prove that $F \equiv G$.

Problem 3 (*). How many points of intersection can a circle have with a parabola?

Problem 4. Construct a first-degree polynomial $P(x)$ such that $P(1) = 1$ and $P(3) = 4$.

Problem 5. Construct a second-degree polynomial $P(x)$ such that $P(1) = 2$, $P(3) = 4$, and $P(5) = 36$.

Lagrange Interpolation

Problem 6. Construct a polynomial $P(x)$ that takes the value 0 at points $x_1, x_2, \ldots, x_k$, and is non-zero at all other points. What is its degree?

Problem 7. Construct a first-degree polynomial $P(x)$ such that $P(x_0) = 1$ and $P(x_1) = 0$.

Problem 8. Construct a second-degree polynomial $P(x)$ such that $P(x_0) = 1$ and $P(x_1) = P(x_2) = 0$.

Problem 9. Construct a degree-n polynomial $P(x)$ such that $P(x_0) = 1$ and $P(x_1) = P(x_2) = \ldots = P(x_n) = 0$.

Problem 10. Construct a degree-n polynomial $P(x)$ such that $P(x_0) = y_0$ and $P(x_i) = 0$ for $0 \leq i \leq n$.

Problem 11. Construct a polynomial $P(x)$ of degree at most n such that $P(x_i) = y_i$ for $0 \leq i \leq n$.

A polynomial of the form

$$P(x) = \sum_{k=0}^{n} y_k \frac{(x - x_0) \cdots (x - x_{k-1})(x - x_{k+1}) \cdots (x - x_n)}{(x_k - x_0) \cdots (x_k - x_{k-1})(x_k - x_{k+1}) \cdots (x_k - x_n)}$$

that takes the values y_i at points x_i is called the Lagrange interpolation polynomial.

Question: How many interpolation polynomials can exist?

Newton Interpolation

Problem 12. Prove that any quadratic trinomial can be written in the form $a + bx + cx(x-1)$.

Problem 13. Find the equation of a parabola passing through the points $(0, 1)$, $(1, 2)$, and $(2, 4)$.

Problem 14. Prove that any cubic polynomial can be written in the form $a + bx + cx(x-1) + dx(x-1)(x-2)$.

Problem 15. Find a cubic polynomial $P(x)$ such that $P(0) = 1$, $P(1) = 2$, $P(2) = 4$, and $P(3) = 8$.

Problem 16. Prove that the polynomial

$$P_n(x) = \frac{x(x-1)(x-2)\cdots(x-n+1)}{n!}$$

takes integer values for all integer x.

Problem 17. a) Prove that if a polynomial takes integer values for all integer x, then it can be written as a sum of polynomials $P_0(x) = 1$, $P_1(x) = x$, $P_2(x) = \frac{x(x-1)}{2!}$, ..., $P_n(x) = \frac{x(x-1)(x-2)\cdots(x-n+1)}{n!}$, with integer coefficients.

b) Prove that this representation is unique.

Additional Exercises

Problem 18. The numbers a, b, and c are distinct. How many roots can the equation

$$\frac{(x-a)(x-b)}{(c-a)(c-b)} + \frac{(x-b)(x-c)}{(a-b)(a-c)} + \frac{(x-a)(x-c)}{(b-a)(b-c)} = 1$$

have?

Problem 19. Prove that for any distinct numbers $a_1, a_2, \ldots, a_n$ and any numbers $b_1, b_2, \ldots, b_n$, there exists a unique polynomial $P(x)$ of degree less than n such that $P(a_1) = b_1$, $P(a_2) = b_2$, ..., $P(a_n) = b_n$.

Problem 20. Find the polynomial of the smallest degree such that when divided by $(x-1)$ it leaves a remainder of 1, when divided by $(x-2)$ it leaves a remainder of 3, and when divided by $(x-4)$ it leaves a remainder of 5.