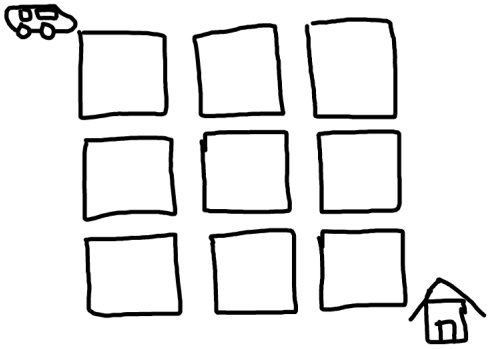
COMBINATORICS

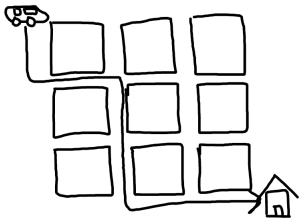
BEGINNER CIRCLE 1/13/2013

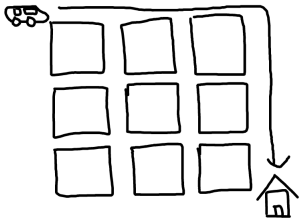
1. The Taxicab Driver

In downtown Los Angeles, a taxi-cab driver has to drive a person home. As you may know, the streets in downtown L.A. are arranged in a grid, like below:



The taxicab driver is allowed to take any path he wishes to take the person home, as long as the path is as short as possible. For instance, here are two different paths that he could take:

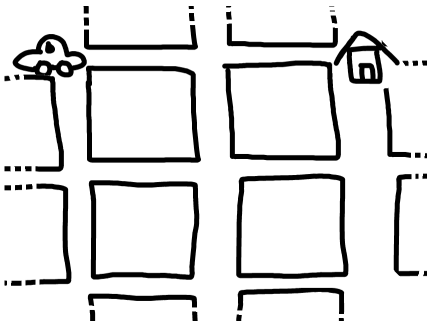




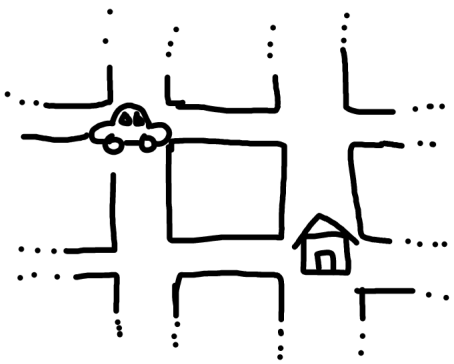
The question, of course, is how many ways can the taxicab driver take his passenger home?

Jeff Hicks/Los Angeles Math Circle/UCLA Department of Mathematics .

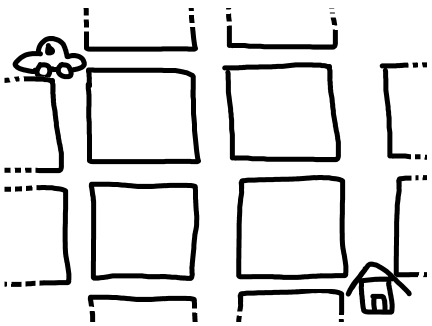
Problem 1. How many ways can the taxicab driver take a man home who lives only two blocks away?



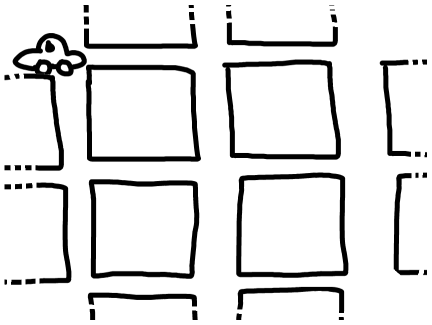
Problem 2. How many ways home are there this time?



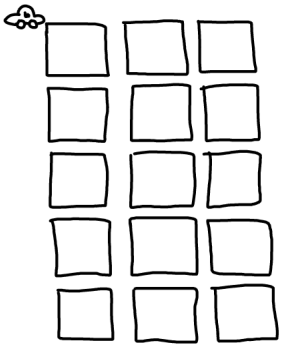
Problem 3. What about in this case?



Problem 4. Can you place 2 houses on this grid, so that they are the same distance (in blocks) away from the taxi, but they have a different number of paths that the taxicab can take to them?

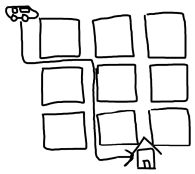


Problem 5. Can you find a house that has 3 possible paths to it? What about 4? What about 5?



Problem 6. Suppose that you are giving a taxi driver directions to your house, which is always East and South of your current position. You could give him directions in

the form of commands of going South and East, so



would be written

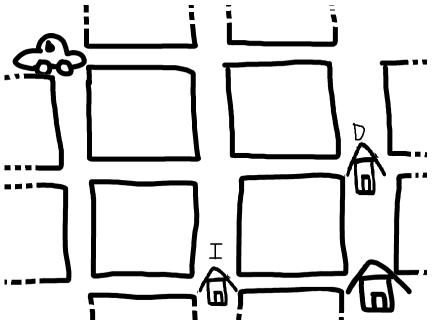
South East South South East

or maybe

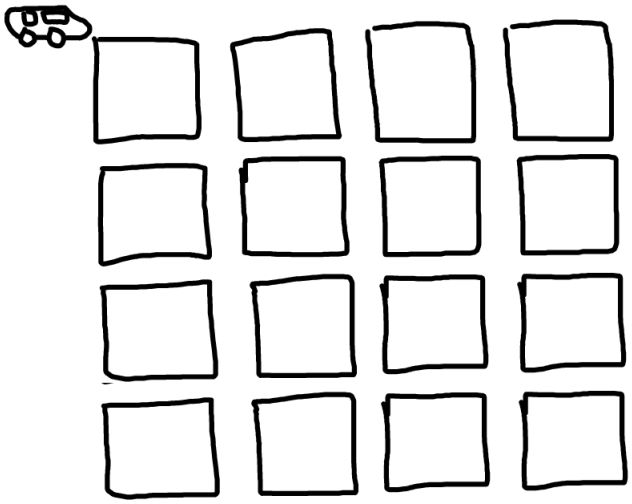
$$SESSE$$

for short. Now, every set of directions from the taxi position to the house will involve exactly 3 souths, and exactly 2 easts. How many ways can you give directions back home?

Problem 7 (*)**.** As Jonathan drive home, he notices he always either passes Isaac's or Derek's house on his journey. He knows from previous experience that there are 3 paths to Isaac's house, and 3 paths to Derek's House. How many paths are there to Jonathan's House?



Problem 8 (*)**.** With the previous problem in mind, find the number of paths to every street intersection in this small city:



2. CHOOSE!

In this section, we ask how many ways we can choose couple of items from a large number of items. For example, let us say that I was running a presentation for the class (which has 20 people), and I needed 3 volunteers. How many ways could I select those volunteers? Let's call the number of ways I could select those volunteers

$$20 \text{ choose } 3$$

meaning the number of ways of choosing 3 items from a set of 20. Note that when I choose items, it doesn't matter what order I choose them in!
To make these problems easier to write, I will write

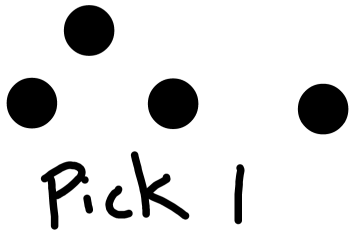
$$\binom{n}{k} = n \text{ choose } k$$

I mean the number of ways to choose k objects from a group of n objects. For example,

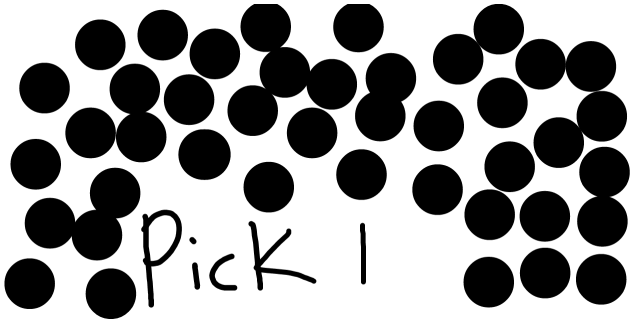
$$\binom{4}{2} = 6$$

Let's look at some simple choosing problems.

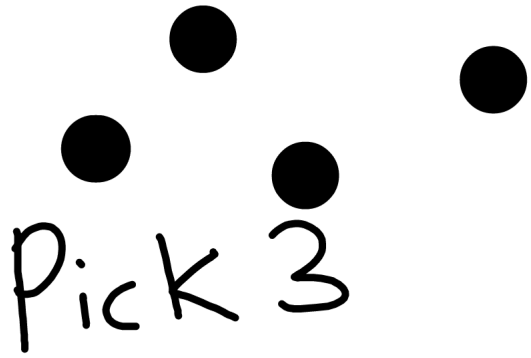
Problem 9. What is the number of ways to choose 1 item from a group of 4 items– or in the math notation, $\binom{4}{1}$?



Problem 10. What is the number of ways to choose 1 item from a group of 42 items– in math notation, $\binom{42}{1}$?

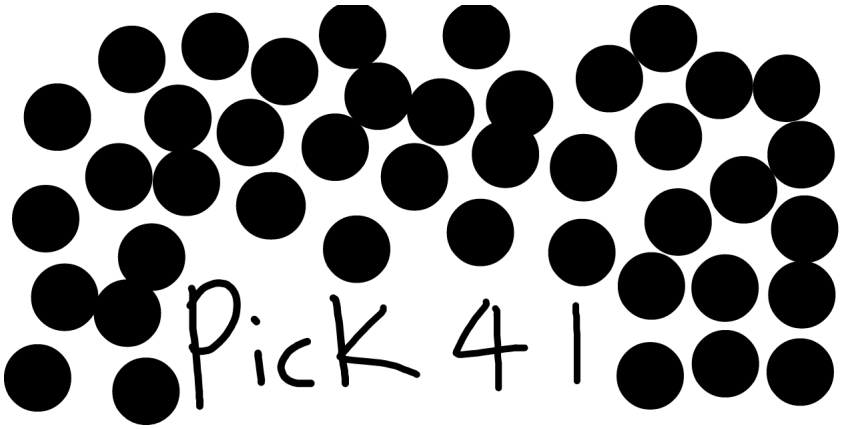


Problem 11. What is the number of ways to choose 3 items from a group of 4 items?



Problem 12. What is the number of ways to choose 41 items from a group of 42 items, or in math talk,

$$\binom{42}{41}$$



Problem 13 (*)**.** What is the value of

$$\binom{n}{1}$$

What about

$$\binom{n}{n-1}$$

Provide a justification for your answer in complete sentences.

Problem 14. What is the value of

(a) $\binom{4}{2}$

(b) $\binom{5}{2}$

(c) $\binom{5}{3}$

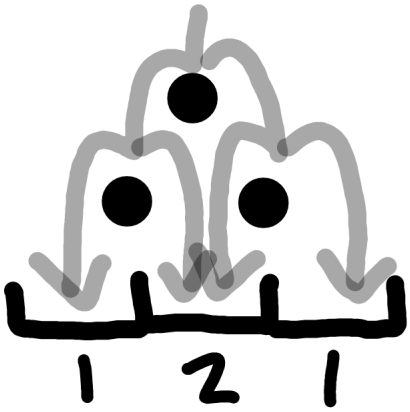
Problem 15 (*)**.** Suppose that your teacher told you that there were 1140 ways to choose 3 students from your class, which is 20 people large.

$$\binom{20}{3} = 1140$$

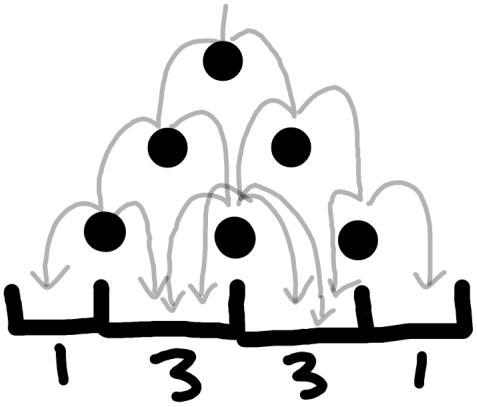
Suppose that you wanted to select 17 students from your class. How many ways could you do that? Justify your answer with a complete sentence.

3. Pachinko and Pascal

A Pachinko machine is a game like pinball. Little balls fall out of the top of the machine, and then fall onto a triangular grid of pins where they bounce until the reach the bottom, where they fall into little bins. Let us look at a simple pachinko machine.

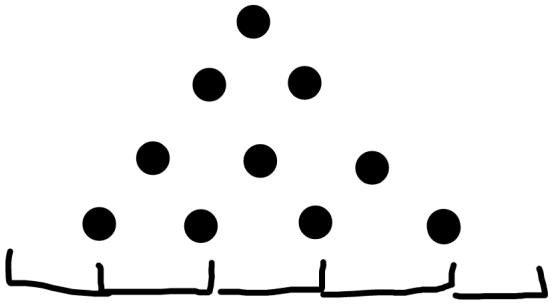


As the ball falls down, there is one possible way for the ball to fall into the 2 side pockets, and 2 possible ways for the ball to fall into the middle pocket.
The pachinko machine above is called a pachinko machine with 2 rows of pins. As we add more rows of pins, there may be more ways for the balls to reach the bottom pins. For example, look at this pachinko machine with 3 rows of pins:

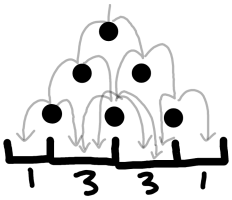


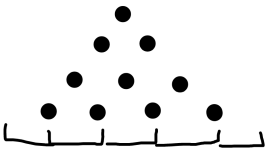
The numbers at the bottom of each bin tell you how many ways that a ball can fall into the selected bin.

Problem 16. Fill out the numbers that say how many ways balls can fall into each bin for this pachinko machine:



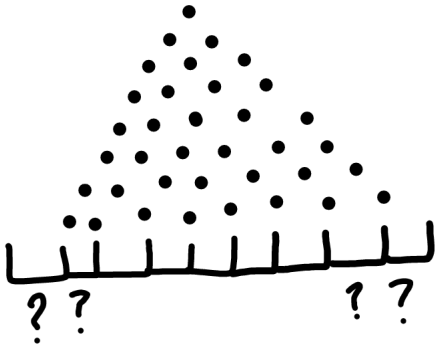
Problem 17. For each of these pachinko machines, sum the values that appear below the bins. What does this sum represent? Do you see a pattern? Answer in full sentences.



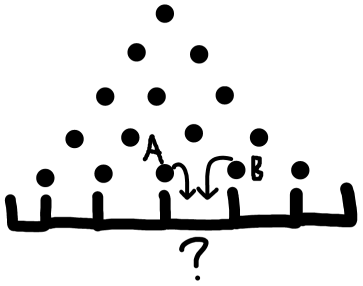


Problem 18. How many ways can the ball fall into the left most bin (no matter how tall the pachinko machine is) . What about the right most bin?

Problem 19. Fill in the question marks for this 8-level pachinko machine



Problem 20 (*)**.** Try to compute the number of ways that a ball can fall into the labeled bin on this pachinko machine.



(a) How many ways can a ball reach the pin labeled A. (Hint! Use a previous problem)

(b) How many ways can a ball reach the pin labaled B?

(c) Every ball that travels to the bin labeled **?** must either travel through A or B, but cannot go through both. How many ways can a ball reach **?**

Problem 21 (Relating Pachinko and Choose)**.** Think about a ball falling to the bottom of a pachinko machine. At every level, it chooses to fall down the left side of the pin or right side of the pin.

(a) Compute $\binom{5}{3}$.

(b) In the above problem, to fall from the top of the pachinko machine into the pin labeled **?**, how many times would the ball need to go right?

(c) Each path from the top of the pachinko machine to the **?** is a choice of 3 rights out of 5 levels. Using the language number of ways to choose traveling right, what would the number of paths to the pin labaled A be called?

(d) The idea that the number of paths to a bin being equal to the number of ways to choose rights is given by the formula

$$\text{Number of paths to a bin} = \binom{\text{number of levels in the pachinko machine}}{\text{number of rights the ball has to make to get to the bin}}$$

With this formula, and the previously drawn pachinko machines, compute
(a) $\binom{5}{2}$

(b) $\binom{4}{2}$

(c) $\binom{3}{1}$

(d) $\binom{5}{4}$

Problem 22 (Bonus 1)**.** Every ball that falls into a bin, either falls from the bin immediatly above and left of it, or above and right of it. How does this (and the problem above) show that

$$\binom{n}{r} = \binom{n-1}{r-1} + \binom{n-1}{r}$$

Answer in complete sentences. (Hint! Let n be the number of levels in a pachinko machine. What should r be?)

Problem 23 (Bonus 2)**.** Suppose that a pachinko machine has n levels to it.

(a) How many ways can a ball fall to the bottom of the pachinko machine? Answer in full sentences(Hint! How many choice can the ball make at each level)

(b) Find the value of the sum

$$\binom{5}{0} + \binom{5}{1} + \binom{5}{2} + \binom{5}{3} + \binom{5}{4} + \binom{5}{5}$$

(c) What about the value of the sum

$$\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \dots \binom{n}{n-1} + \binom{n}{n}$$

Explain your answer in full sentences.