Geometric Optimization

Prepared by Mark on November 3, 2024
Based on a handout by Nakul & Andreas

Instructor's Handout

This file contains solutions and notes.
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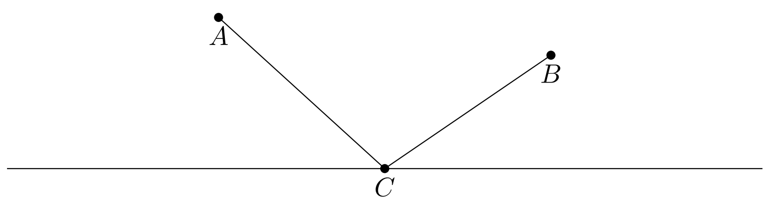
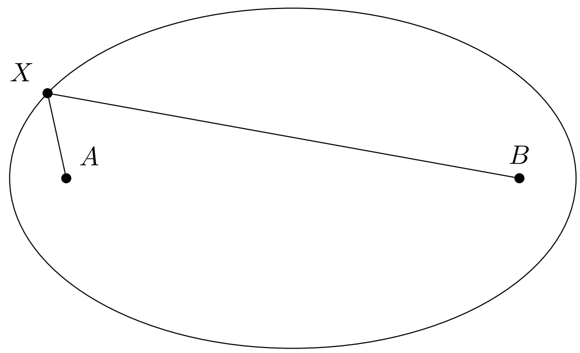
Part 1: Optimization

Problem 1:
Let A and B be two points on the same side of a given line ℓ.
Find a point C on ℓ so that $|AC| + |BC|$ is minimized.



Definition 1:
An *ellipse* with foci A, B and radius r is the set of all points C where $|AB| + |BC| = r$.

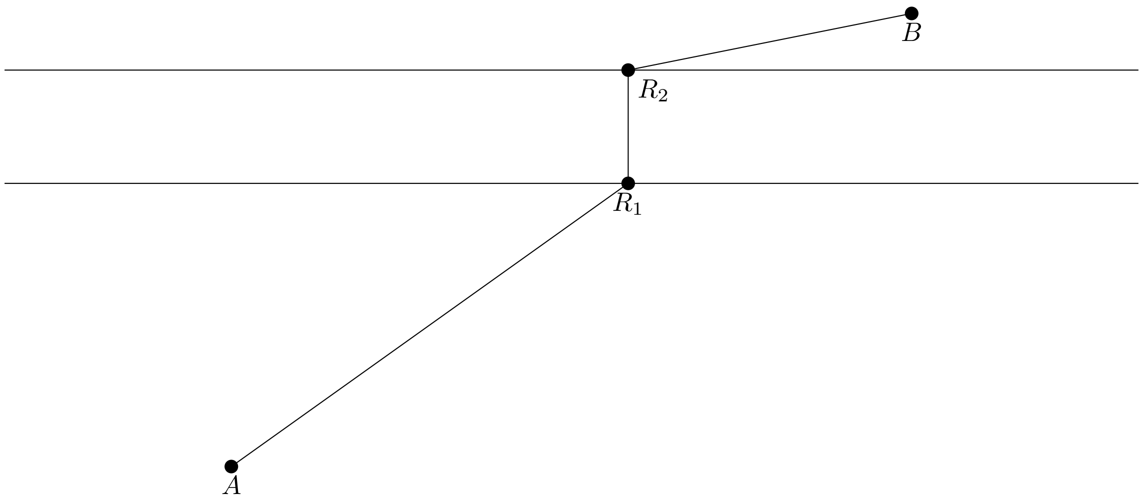
Problem 2:
Consider a reflective ellipse with foci A and B.
Find all points X on the ellipse where A can aim a laser at so that the beam reaches B.
Hint: use Problem 1



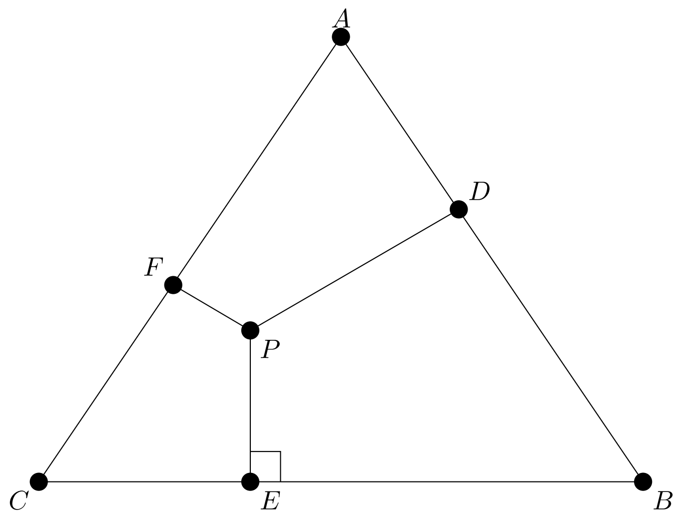
Problem 3:
Let C be a point in the interior of a given angle. Find points A and B on the sides of the angle such that the perimeter of the triangle ABC is a minimum.

Problem 4:
In a convex quadrilateral ABCD, find the point T for which the sum of the distances to the vertices is minimal.

Problem 5:
A road needs to be constructed from town A to town B, crossing a river, over which a perpendicular bridge is to be constructed. Where should the bridge be placed to minimize $|AR_1| + |R_1R_2| + |R_2B|$?



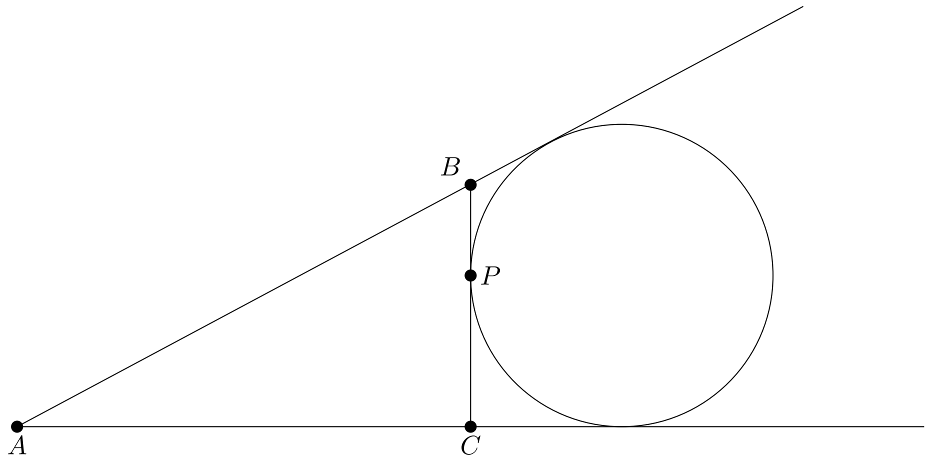
Problem 6:
Consider an equilateral triangle triangle with vertices labeled A, B, and C.
Let P be a point inside this triangle. Place D, E, and F so that PD, PE, and PF are the perpendiculars from P to the sides of the triangle.
Find all points P where $|PD| + |PE| + |PF|$ is minimized.



Problem 7:
With the same setup as Problem 6, find all points P where $|PA| + |PB| + |PC|$ is minimized.

Problem 8:
Solve Problem 7 for a triangle that isn't equilateral.

Problem 9:
Draw a circle, then draw two distinct tangents ℓ_1 and ℓ_2 that intersect at point A.
Let P be a point on the circle between the tangents, and BC be the tangent at that point. Describe how P shoud be selected in order to minimize the perimeter of triangle ABC.



Problem 10:
Now, assume that ℓ_1 and ℓ_2 intersect at A, and pick a point P between them.
Find BC through P so that the perimeter of ABC is minimized.

Part 2: Bonus Problems

Problem 11:
Given a cube $A_1B_1C_1D_1A_2B_2C_2D_2$ with side length l,
find the angle and distance between lines A_1B_2 and A_2C_1.

Solution

Triangle $A_1B_2D_2$ is equilateral.
Also, point A_2 is equidistant from each of this triangle's vertices.
Therefore, its projection onto the plane formed by A_1, B_2, and D_2 is the center of the triangle.

Similarly, C_1 is mapped to the center of $A_1B_2D_2$.
Therefore, lines A_1B_2 and A_2C_1 are perpendicular and the distance between them is equal to the distance from the center of triangle $A_1B_2D_2$ to its side.

Since all the sides of this triangle have length $l\sqrt{2}$, the distance in question is $\frac{a}{\sqrt{6}}$.

Problem 12:
Consider a cube $A_1B_1C_1D_1A_2B_2C_2D_2$, and let K, L,
and M be midpoints of the edges A_2D_2, A_1B_1, and C_1C_2.
Show that the triangle formed by KLM is equilateral, and that its center is the center of the cube.

Solution

Let O be the center of the cube. Then, $|OK| = |C_1D_2|$, $|2OL| = |D_2A_1|$, and $2|OM| = |A_1C_1|$.
Since triangle $C_1D_2A_1$is equilateral, triangle KLM is equilateral and has O as its center.

Problem 13:
Consider all n-gons with a certain perimeter. Show that the n-gon with maximal area has equal sides

Problem 14:
Consider all n-gons with a certain perimeter. Show that the n-gon with maximal area has equal angles