

AMC Problem Session Answers  
 UCLA Math Circle  
 01/13/2013

1. (AMC 1973 #10) Suppose  $n$  is a real number such that the following simultaneous system has no solution. What is the value of  $n$ ?

$$\begin{aligned} nx + y &= 1 \\ ny + z &= 1 \\ x + nz &= 1 \end{aligned}$$

**ANSWER:**  $n = -1$

2. (AMC 1973 #16) The sum of all the angles except one of a convex  $n$ -gon is  $2190^\circ$ . What is  $n$ ?

**ANSWER:** 15

3. (AMC 1973 #19) For positive numbers  $n$  and  $a$ , define  $n_a!$  by

$$n_a! = n(n - a)(n - 2a)(n - 3a) \dots (n - ka),$$

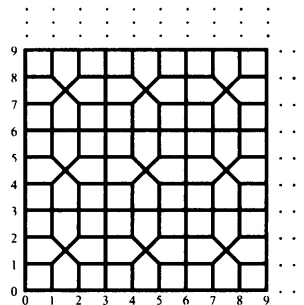
where  $n > ka$  and  $n \leq (k + 1)a$ . Evaluate the quotient  $72_8! / 18_2!$ .

**ANSWER:**  $4^9$

4. (AMC 1973 #33) When one ounce of water is added to a mixture of milk and water, the new mixture is 20% milk. When one ounce of milk is added to this new mixture, the result is  $1/3$  milk. What is the percentage of milk in the original mixture?

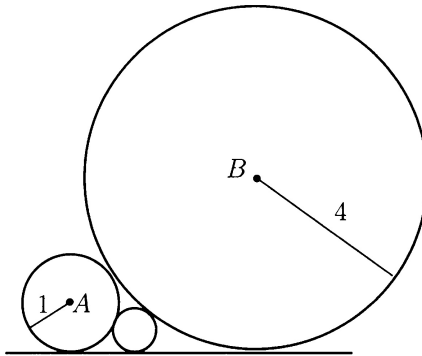
**ANSWER:** 25%

5. (AMC12 2001 #10) The plane is tiled by congruent squares and congruent pentagons as in the picture below. What is the percentage of the plane that is covered by the pentagons?



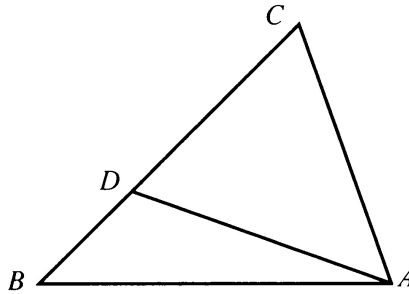
**ANSWER:**  $5/9$

6. (AMC12 2001 #18) A circle centered at  $A$  with radius 1 and a circle centered at  $B$  with radius 4 are externally tangent. A third circle is tangent to the first two and to one of their common external tangents as in the picture below. What is the radius of the third circle?



**ANSWER:**  $4/9$ . One can use Descartes' formula  $2(a^2 + b^2 + c^2 + d^2) = (a + b + c + d)^2$ , with  $a, b, c, d$  the reciprocals of the radii of four externally tangent circles.

7. (AMC12 2001 #24) In triangle  $ABC$ ,  $\angle ABC = 45^\circ$ . Point  $D$  is on  $\overline{BC}$  so that  $2 \cdot BD = CD$  and  $\angle DAB = 15^\circ$ . What is the measure of  $\angle ACB$ ?



**ANSWER:**  $75^\circ$

8. (AMC12B 2002 #25) Let  $f(x) = x^2 + 6x + 1$ , and  $P$  denote the set of points  $(x, y)$  in the coordinate plane such that

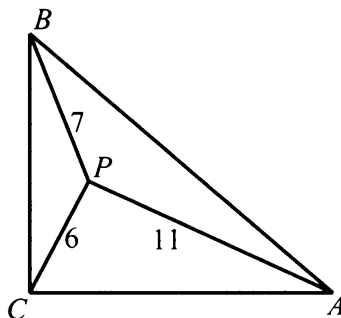
$$f(x) + f(y) \leq 0 \text{ and } f(x) - f(y) \leq 0.$$

What is the area of this region? **ANSWER:**  $8\pi$

9. (AMC12B 2007 #22) Two ants walk along the edges of an equilateral  $\triangle ABC$  with the same speed and direction. A third ant walks inside  $\triangle ABC$  such that it is always on the midpoint of the segment connecting the first two ants. Let  $T$  be the region enclosed by the trajectory of the third ant. What is the ratio between the area of  $T$  and the area of  $\triangle ABC$  if the first two ants start at the point  $A$  and the midpoint of  $BC$  respectively?

**ANSWER:**  $1/16$

10. (AMC12B 2006 #23) Isosceles  $\triangle ABC$  has a right angle  $C$ . Point  $P$  is inside  $\triangle ABC$  such that  $PA = 11$ ,  $PB = 7$  and  $PC = 6$ . What is the area of  $\triangle ABC$ ?



**ANSWER:**  $21\sqrt{2} + \frac{85}{2}$

11. (AMC 12B 2007 #16) Each face of a regular tetrahedron is painted either red, white, or blue. Two colorings are considered the same if one can be made into another by rotating the tetrahedron. How many different colorings are there?

**ANSWER:** 15. By Burnside's Lemma,  $|X/G| = \frac{1}{|G|} \sum_{g \in G} |X^g|$ .

12. (AMC12 2001 #11) A box contains exactly five chips, 3 red and 2 white. Chips are randomly removed one at a time without replacement until all the red chips are drawn or all the white chips are drawn. What is the probability that the last chip drawn is white?

**ANSWER:**  $\frac{3}{5}$ . In general, the answer is  $\frac{r}{r+w}$  if there are  $r$  red chips and  $w$  white chips.

13. Let  $S = \{1, 2, \dots, 2002\}$ . A 15-element subset of  $S$  is called *excellent* if the sum of its elements is divisible by 7. Find the number of excellent subsets of  $S$ .

**ANSWER:**  $\frac{1}{7} \binom{2002}{15}$  by symmetry.

14. (AIME II 2006 # 4) Let  $(a_1, a_2, a_3, \dots, a_{12})$  be a permutation of  $(1, 2, 3, \dots, 12)$  for which

$$a_1 > a_2 > a_3 > a_4 > a_5 > a_6 \text{ and } a_6 < a_7 < a_8 < a_9 < a_{10} < a_{11} < a_{12}.$$

Find the number of such permutations.

**ANSWER:**  $\binom{11}{5} = 462$ .

15. Find the number of unordered triples of non-negative integers  $(a, b, c)$  such that  $a + b + c = 2012$ .

**ANSWER:** 338352. In general for  $a + b + c = n$ , the answer is  $\frac{1}{6} \left( \binom{n+2}{2} + \lceil \frac{n+1}{2} \rceil \cdot 3 + \delta \right)$ , where  $\delta$  is 2 when  $3|n$  and 0 otherwise.